Selective Quantum Operations

Definition 1 (*selective quantum operation*). A *selective quantum operation* (SQO) [Cle97] is a family of $s \times r$ matrices A_{i,j_i} $(1 \le i \le m,\ 1 \le j_i \le k_i)$:

$$\begin{array}{l} A_{1,1},\ A_{1,2},\ \ldots,\ A_{1,k_1} \\ A_{2,1},\ A_{2,2},\ \ldots,\ A_{2,k_2} \\ \ldots \\ A_{m,1},\ A_{m,2},\ \ldots,\ A_{m,k_m} \end{array}$$

with entries in $\mathbb{C}$, such that

$$\sum_{i=1}^{m} \sum_{j=1}^{k_i} A_{i,j}^{\dagger} A_{i,j} = \mathbb{1}_{r \times r}.$$

The integer r is the *input dimension* and s is the *output dimension*. Let

$$B_i := \sum_{j=1}^{k_i} A_{i,j}^{\dagger} A_{i,j}$$

for $1 \le i \le m$. On input ρ, where ρ is an $r \times r$ density matrix, the above SQO will yield the the pair

$$\left(\frac{1}{\mathrm{tr}(\rho B_i)} \sum_{j=1}^{k_i} A_{i,j} \rho A_{i,j}^{\dagger}\ ,\ \ i \right)$$

with probability $\mathrm{tr}(\rho B_i)$. The left-hand-side member of the pair is the *the quantum residue*, an $s \times s$ density matrix, and the right-hand-side member i is the *classical output* of the SQO. ▲

Definition 2 (*superoperator*). An r-dimensional *superoperator*, also known as a *trace-preserving quantum operation*, is a special case of a SQO where

$$\begin{array}{rcl} s & := & r, \\ m & := & 1, \\ k & := & k_1, \\ A_j & := & A_{1,j} \quad (\text{for } 1 \le j \le k). \end{array}$$

Accordingly, the family of matrices $\{A_j\}$ is such that

$$\sum_{j=1}^{k} A_j^{\dagger} A_j = \mathbb{1}.$$

The family is called the *operator-sum representation* [NC00, page 360] of the superoperator, and the individual matrices A_j are called *operation elements*. On input ρ, a density matrix, this superoperator will produce the state

$$\sum_{j=1}^{k} A_j \rho A_j^{\dagger}.$$

▲

Definition 3 (*POVM*). An r-dimensional *POVM*, also known as a *positive operator-valued measure*, is a special case of a SQO where

$$s := r,$$

and for $1 \leq i \leq m$:

$$\begin{aligned} k_i &:= 1, \\ A_i &:= A_{i,1}, \\ B_i &:= A_i^\dagger A_i. \end{aligned}$$

Accordingly, the families of matrices $\{A_i\}$ and $\{B_i\}$ are such that

$$\sum_{i=1}^{m} A_i^\dagger A_i = \sum_{i=1}^{m} B_i = \mathbb{1}.$$

The positive matrices B_i are called the *POVM elements* [NC00, page 90]. On input ρ, a density matrix, this POVM will output the classical value i, with probability $\mathrm{tr}(\rho B_i)$, together with the quantum residue

$$\frac{1}{\mathrm{tr}(\rho B_i)} A_i \rho A_i^\dagger.$$

▲

Definition 4 (*projective measurement*). An r-dimensional *projective measurement*, also known as a *von Neumann measurement*, is a special case of a POVM where $P_i := A_i$ is a projection operator for $1 \leq i \leq m$. Accordingly, we have

$$\begin{gathered} P_i = A_i = B_i = P_i^\dagger = P_i^2 \quad \text{for } 1 \leq i \leq m, \\ P_{i_1} P_{i_2} = 0 \quad \text{if } i_1 \neq i_2, \\ \sum_{i=1}^{m} P_i = \mathbb{1}. \end{gathered}$$

The family $\{P_i\}$ is called a *complete set of orthogonal projectors*. On input ρ, a density matrix, this projective measurement will output the classical value i, with probability $\mathrm{tr}(\rho P_i)$, together with the quantum residue

$$\frac{1}{\mathrm{tr}(\rho P_i)} P_i \rho P_i.$$

▲

References

[Cle97] Richard Cleve. Selective quantum operations. Two pages photocopied manuscript, contact Paul Dumais `dumais@iro.umontreal.ca`, June 1997.

[NC00] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information.* Cambridge University Press, Cambridge, United Kingdom, 2000.