(1) DEFINITIONS. Let $A = [a_{i,j}] \in \mathbb{C}^{m \times n}$, then

$$A \otimes B := [a_{i,j} B]$$

$$\mathrm{Vec}(A) := [a_{1,1} \ldots a_{m,1}\ a_{1,2} \ldots a_{m,2}\ \ \ldots\ \ a_{1,n} \ldots a_{m,n}]^{\mathsf{T}}$$

(2) $\exists\, A, B:\ A \otimes B \neq B \otimes A$

(3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

(4) [1] $(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$

(5) $\lambda A \otimes \mu B = \lambda\mu(A \otimes B)$

(6) [1] $(A \otimes B)(C \otimes D) = AC \otimes BD$

(7) $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

(8) $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$

(9) Let $\{V_1, V_2, \ldots, V_k\}$ be a lin. ind. set of non-null matrices.

[1] $$\sum_{j=1}^{k} A_j \otimes V_j = 0 \quad \Leftrightarrow \quad \forall\, j:\ A_j = 0$$

(10) [1] $\mathrm{Vec}(A + B) = \mathrm{Vec}(A) + \mathrm{Vec}(B)$

(11) $\mathrm{Vec}(\lambda A) = \lambda\,\mathrm{Vec}(A)$

(12) [1] $\mathrm{Vec}(ABC) = (C^{\mathsf{T}} \otimes A)\mathrm{Vec}(B)$

(13) [1] $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$

(14) $\mathrm{tr}(\lambda A) = \lambda\,\mathrm{tr}(A)$

(15) [1] $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

(16) [1] $\mathrm{tr}(AB) = (\mathrm{Vec}(A^{\mathsf{T}}))^{\mathsf{T}}\mathrm{Vec}(B)$

(17) DEFINITIONS.

U is a *unitary* operator if $U^\dagger U = UU^\dagger = \mathbb{1}$.

H is a *hermitian* operator if $H^\dagger = H$.

P is a *projection* operator if $P^\dagger = P$ and $P^2 = P$.

Q is a *positive* operator if $\exists\, S:\ Q = S^\dagger S$.

ρ is a *density* operator if ρ is positive and $\mathrm{tr}(\rho) = 1$.

N is a *normal* operator if $N^\dagger N = NN^\dagger$.

(18) N is normal $\Leftrightarrow$ N is unitarily diagonalizable.

(19) H is hermitian $\Rightarrow \mathrm{e}^{\mathrm{i}H}$ is unitary.

(20) Let λ be an eigenvalue of operator A.

A is unitary $\Rightarrow |\lambda| = 1$.

A is hermitian $\Rightarrow \lambda \in \mathbb{R}$.

A is a projection $\Rightarrow \lambda \in \{0, 1\}$.

A is positive $\Rightarrow \lambda \geq 0$.

A is a density operator $\Rightarrow 0 \leq \lambda \leq 1$.

(21) Let $\{|\theta_j\rangle\}_{1\le j\le n}$ be any orthonormal basis of $\mathbb{C}^n$, then

$$\sum_{j=1}^{n} |\theta_j\rangle\langle\theta_j| = \mathbb{1}$$

[1]

$$\mathrm{tr}(A) = \sum_{j=1}^{n}\langle\theta_j|A|\theta_j\rangle$$

(22) [1,2] (*Cauchy-Schwartz*) $\forall\, \vec{u},\vec{v}:\ \langle\vec{u},\vec{v}\rangle\langle\vec{v},\vec{u}\rangle \le \langle\vec{u},\vec{u}\rangle\langle\vec{v},\vec{v}\rangle.$

(23) $U \in \mathbb{C}^{n\times n}$ is unitary iff
$\exists\, \{|\theta_j\rangle\},\ \{|\theta'_j\rangle\}$, orthonormal bases of $\mathbb{C}^n$:

$$U = \sum_{j=1}^{n} |\theta_j\rangle\langle\theta'_j|$$

(24) (*Pythagoras*) Let $\{P_j\}_{1\le j\le k}$ be projectors of $\mathbb{C}^n$, then

$$P_iP_j = 0 \text{ for } i \ne j \text{ and } \sum_{j=1}^{k}\mathrm{tr}(P_j) = n \ \Leftrightarrow\ \sum_{j=1}^{k} P_j = \mathbb{1}.$$

(25) [1,2] Q is positive $\Leftrightarrow$ Q is hermitian and $\forall\, |\psi\rangle :\ \langle\psi|Q|\psi\rangle \ge 0.$

(26) [2] $\rho \in \mathbb{C}^{n\times n}$ is a density operator iff
$\exists\, \{w_j\},\ 0 \le w_j \le 1,\ \sum w_j = 1$ and $\{|\phi_j\rangle\}$:

$$\rho = \sum_{j=1}^{n} w_j|\phi_j\rangle\langle\phi_j|.$$

(27) (*spectral decomposition*) $N \in \mathbb{C}^{n\times n}$ is normal iff
$\exists\, \{\lambda_j\},\ \lambda_j \in \mathbb{C}$ and $\{|\theta_j\rangle\}$, an orthonormal basis of $\mathbb{C}^n$:

$$N = \sum_{j=1}^{n} \lambda_j|\theta_j\rangle\langle\theta_j|.$$

(28) (*polar decomposition*) Let $A \in \mathbb{C}^{m\times n}$, then
$\exists\, \{a_j\},\ a_j \ge 0$ and $\{|\theta_j\rangle\},\ \{|\theta'_j\rangle\}$, orthonormal sets of
r vectors of $\mathbb{C}^m$, $\mathbb{C}^n$ respectively :

$$A = \sum_{j=1}^{r} a_j|\theta_j\rangle\langle\theta'_j|.$$

[1]Provided that the dimensions of the matrices involved are compatible.
[2]The *ket* $|\psi\rangle$ notation is reserved for norm-1 vectors while $\vec{u}$ denotes any vector.