Let ρ_1, ρ_2 and ρ be states, let $|\psi\rangle$ be a pure state, let U be a unitary transformation, let $\mathcal{O}$ be a superoperator, all living or operating in the same space.

(F1) DEFINITION (*fidelity*).

$$\mathcal{F}(\rho_1, \rho_2) := \left(\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right)^2$$

(F2) $0 \le \mathcal{F}(\rho_1, \rho_2) \le 1$ and $\mathcal{F}(\rho_1, \rho_2) = 1 \iff \rho_1 = \rho_2$

(F3) $\mathcal{F}(\rho_1, \rho_2) = \mathcal{F}(\rho_2, \rho_1)$

(F4) $\mathcal{F}(\rho, |\psi\rangle\langle\psi|) = \langle\psi|\rho|\psi\rangle$

(F5) (*multiplicativity*)
Let σ_1, σ_2 be two states living in the same space, then
$\mathcal{F}(\rho_1 \otimes \sigma_1, \rho_2 \otimes \sigma_2) = \mathcal{F}(\rho_1, \rho_2)\mathcal{F}(\sigma_1, \sigma_2)$.

(F6) (*maximum over purifications*)
$\mathcal{F}(\rho_1, \rho_2) = \max\{\ |\langle\psi_1|\psi_2\rangle|^2 \ :\ \rho_j^{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}(|\psi_j\rangle^{\mathrm{A,B}}) \text{ for } j = 1, 2\ \}$

(F7) (*invariance under unitary transformation*)
$\mathcal{F}(U\rho_1 U^\dagger, U\rho_2 U^\dagger) = \mathcal{F}(\rho_1, \rho_2)$

(F8) (*monotonicity*)
$\mathcal{F}(\mathcal{O}(\rho_1), \mathcal{O}(\rho_2)) \ge \mathcal{F}(\rho_1, \rho_2)$

(F9) (*strong concavity*)
Let $0 \le p_j, q_j \le 1$, $\sum p_j = \sum q_j = 1$, let $\{\rho_j\}$ and $\{\sigma_j\}$ be families of states living in the same space, all indexed over the same finite set, then

$$\mathcal{F}(\sum p_j \rho_j, \sum q_j \sigma_j) \ge \left(\sum \sqrt{p_j q_j \mathcal{F}(\rho_j, \sigma_j)}\right)^2.$$

(F10) $\mathcal{F}(\rho_1, \rho_2) \ge \mathrm{tr}(\rho_1 \rho_2)$

Let ρ be a state, let U be a unitary transformation, let $\{P_i\}$ be a projective measurement, all living in a d-dimensional hilbert space $\mathcal{H}^{\mathrm{A,B,C}}$. Let $\rho^{\mathrm{A}} := \mathrm{tr}_{\mathrm{B,C}}(\rho)$, $\rho^{\mathrm{B}} := \mathrm{tr}_{\mathrm{A,C}}(\rho)$, $\rho^{\mathrm{A,B}} := \mathrm{tr}_{\mathrm{C}}(\rho)$, $\rho^{\mathrm{A,C}} := \mathrm{tr}_{\mathrm{B}}(\rho)$, $\rho^{\mathrm{B,C}} := \mathrm{tr}_{\mathrm{A}}(\rho)$.

(E1) Definitions (*von Neumann entropy and mutual information*).

$$S(\rho) := \sum_{j=1}^{r} w_j \lg(\frac{1}{w_j}) \text{ where } w_1, w_2, \ldots, w_r \text{ are the non-zero eigenvalues of } \rho,$$

$$S(\rho^{\mathrm{A}} : \rho^{\mathrm{B}}) := S(\rho^{\mathrm{A}}) + S(\rho^{\mathrm{B}}) - S(\rho^{\mathrm{A,B}}).$$

(E2)

$$0 \leq S(\rho) \leq \lg d$$

$$S(\rho) = 0 \iff \rho = |\psi\rangle\langle\psi| \text{ for some pure state } |\psi\rangle$$

$$S(\rho) = \lg d \iff \rho = \frac{1}{d}\mathbb{1}$$

(E3) (*equality over the subsystems of a bipartite pure state*)

If $\rho^{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}(|\psi\rangle\langle\psi|)$ and $\rho^{\mathrm{B}} = \mathrm{tr}_{\mathrm{A}}(|\psi\rangle\langle\psi|)$ for some pure state $|\psi\rangle^{\mathrm{A,B}}$, then

$$S(\rho^{\mathrm{A}}) = S(\rho^{\mathrm{B}}).$$

(E4) (*subadditivity and triangle inequality*)

$$|S(\rho^{\mathrm{A}}) - S(\rho^{\mathrm{B}})| \leq S(\rho^{\mathrm{A,B}}) \leq S(\rho^{\mathrm{A}}) + S(\rho^{\mathrm{B}})$$

$$S(\rho^{\mathrm{A,B}}) = S(\rho^{\mathrm{A}}) + S(\rho^{\mathrm{B}}) \iff \rho^{\mathrm{A,B}} = \rho^{\mathrm{A}} \otimes \rho^{\mathrm{B}}$$

(E5) (*strong subadditivity*)

$$S(\rho^{\mathrm{A}}) + S(\rho^{\mathrm{B}}) \leq S(\rho^{\mathrm{A,C}}) + S(\rho^{\mathrm{B,C}})$$

$$S(\rho^{\mathrm{A,B,C}}) + S(\rho^{\mathrm{B}}) \leq S(\rho^{\mathrm{A,B}}) + S(\rho^{\mathrm{B,C}})$$

(E6) (*invariance under unitary transformation*)

$$S(U\rho U^{\dagger}) = S(\rho)$$

(E7) (*monotonicity under projective measurement*)

$$S(\sum P_i \rho P_i) \geq S(\rho)$$

(E8) (*concavity and von Neumann vs Shannon*)

Let $\{\rho_j\}_{j=1}^{n}$ be a family of states living in the same space, let $0 \leq p_j \leq 1$, $\sum p_j = 1$, then

$$\sum p_j S(\rho_j) \leq S(\sum p_j \rho_j) \leq \sum p_j S(\rho_j) + H(p_1, \ldots, p_n), \text{ and}$$

$$S(\sum p_j \rho_j) = \sum p_j S(\rho_j) + H(p_1, \ldots, p_n) \iff$$

the states ρ_j have support in orthogonal subspaces.

(E9) (*monotonicity of mutual information under trace-preserving operation*)

Let $\mathcal{O}$ be a superoperator acting on space $\mathcal{H}^{\mathrm{A,B}}$,

let $\rho'^{\mathrm{A}} := \mathrm{tr}_{\mathrm{B}}(\mathcal{O}(\rho^{\mathrm{A,B}}))$ and $\rho'^{\mathrm{B}} := \mathrm{tr}_{\mathrm{A}}(\mathcal{O}(\rho^{\mathrm{A,B}}))$, then

$$S(\rho'^{\mathrm{A}} : \rho'^{\mathrm{B}}) \leq S(\rho^{\mathrm{A}} : \rho^{\mathrm{B}}).$$