Introduction to theoretical quantum CRYPTOGRAPHY

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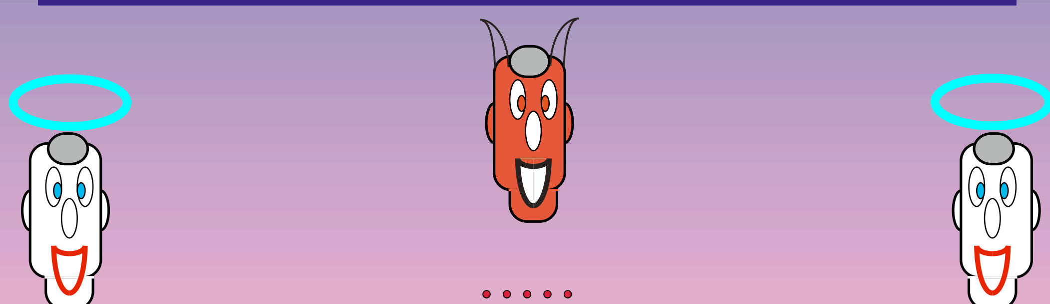
(1) Classical Cryptography

(1.1) Information Theoretical Cryptography

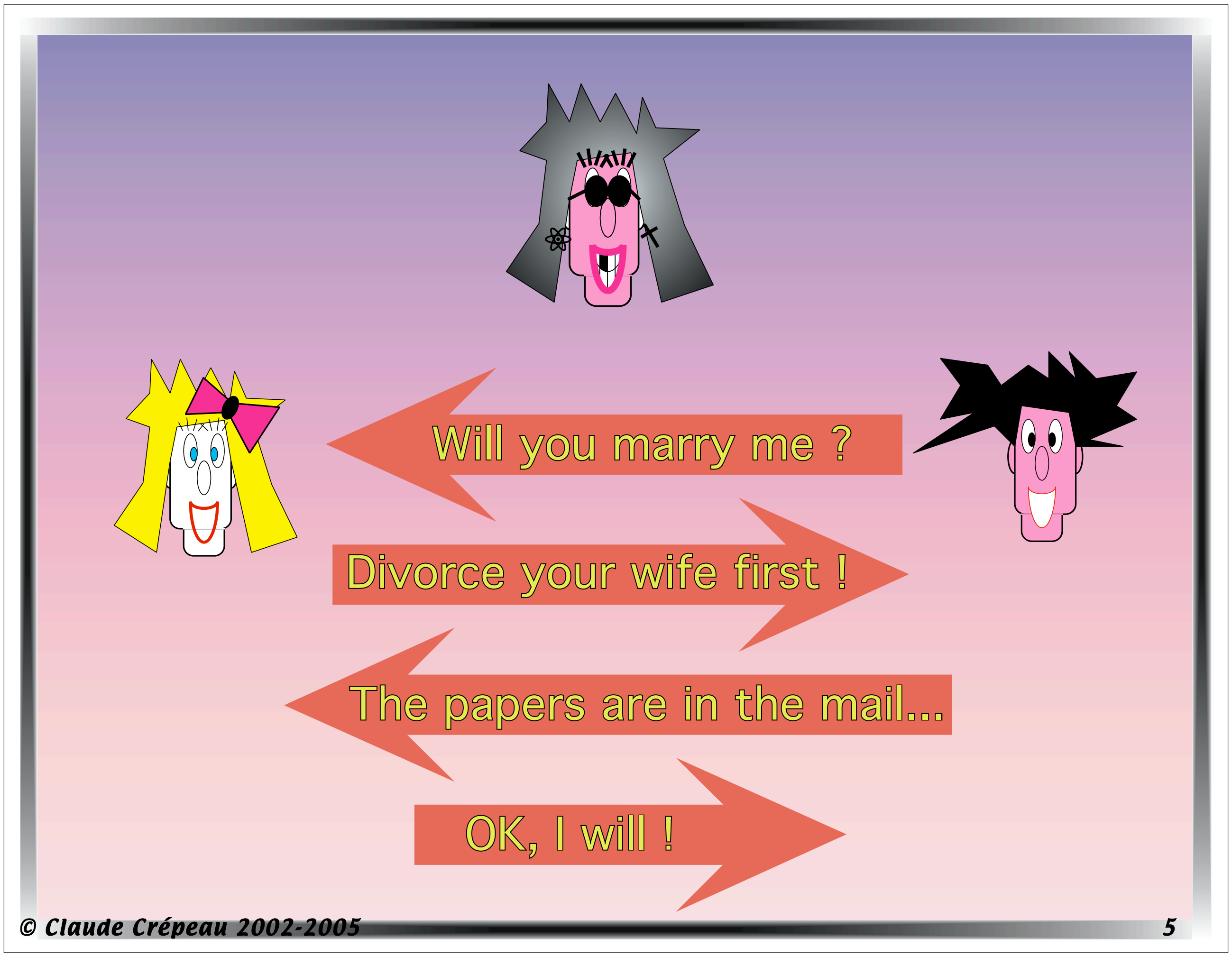
(1.1) Information Theoretical Cryptography



(1.1.1) key distribution

(1.1.2) Encryption

(1.1.3) Authentication


Will you marry me ?
Divorce your wife first !
The papers are in the mail...
OK, I will !
© Claude Crépeau 2002-2005
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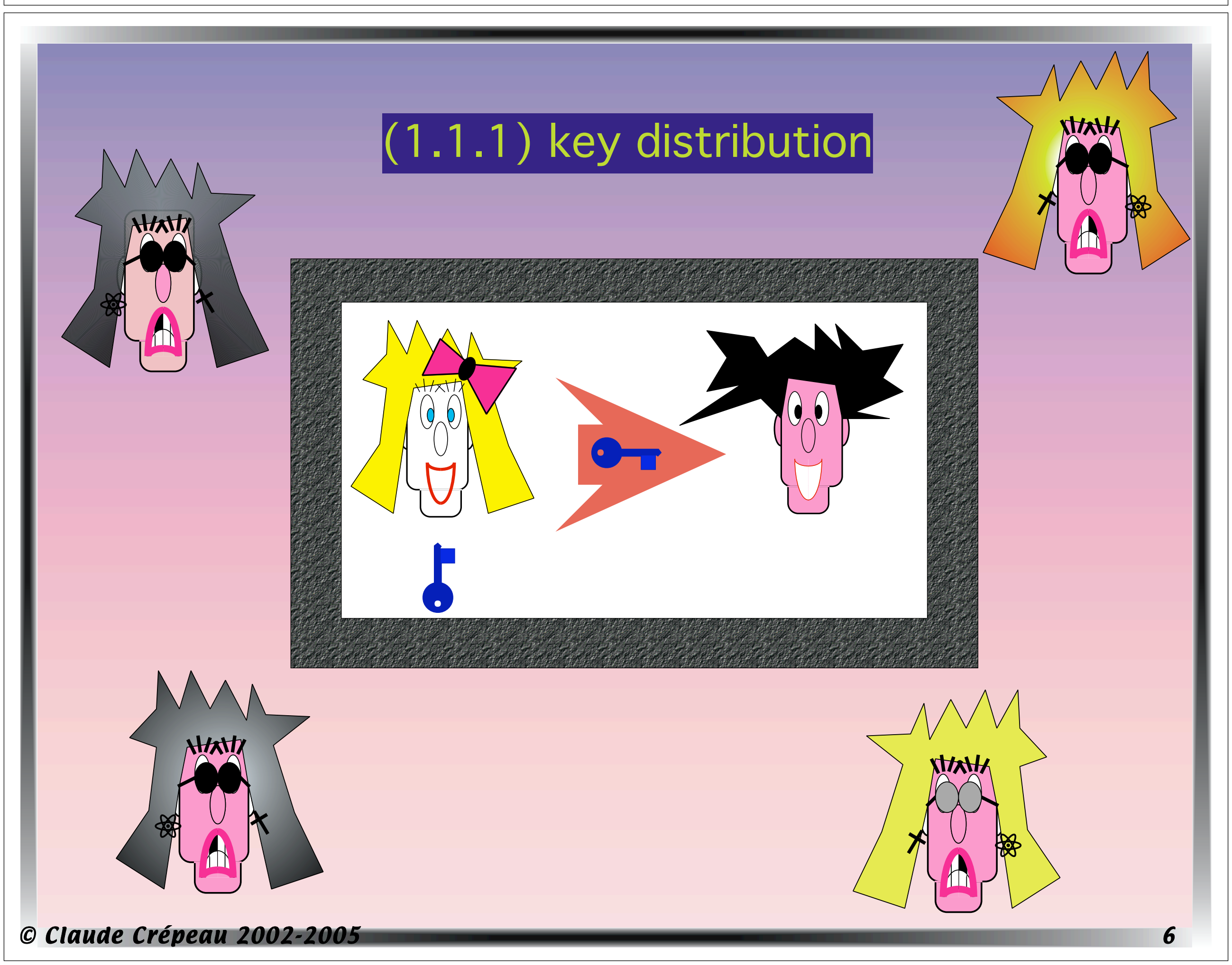
(1.1.1) key distribution
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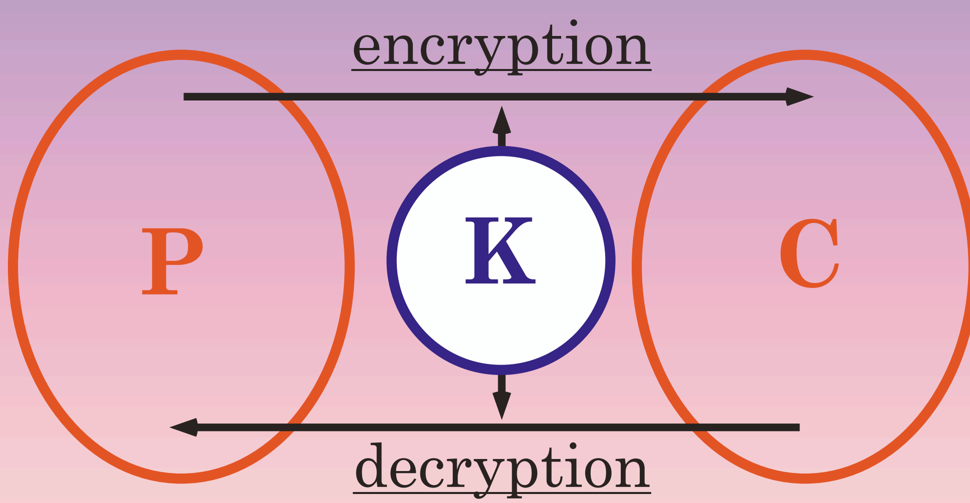
(1.1.2) Encryption

symmetric encryption

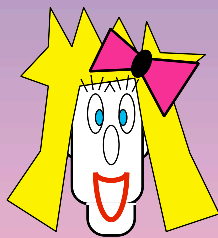


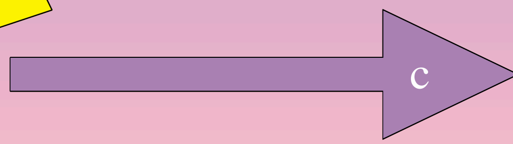
Information Theoretical Security

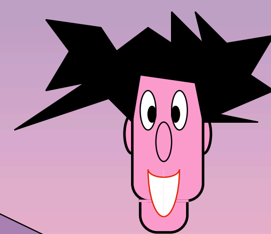
Vernam's One-Time-Pad

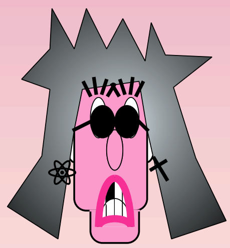
m	⊕ k =	c
1	1	0
0	1	1
1	1	0
0	0	0
0	0	0
1	1	0
0	1	1
0	0	0
1	1	0
1	1	0
1	0	1
1	1	0
1	0	1
0	1	1
0	1	1
1	1	0

⊕ =







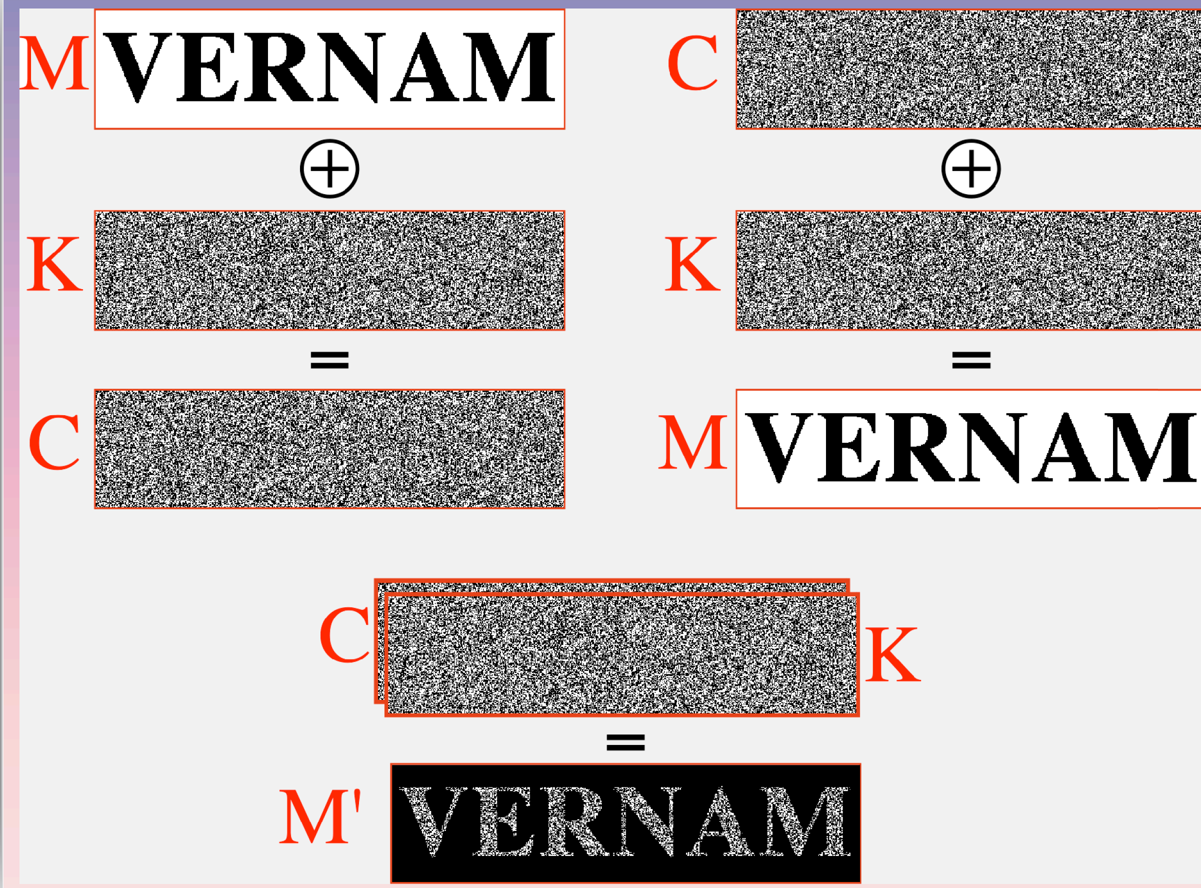


c	⊕ k =	m
0	1	1
1	1	0
0	1	1
0	0	0
0	0	0
0	1	1
1	1	0
0	0	0
0	1	1
0	1	1
1	0	1
0	1	1
1	0	1
1	1	0
1	1	0
0	1	1

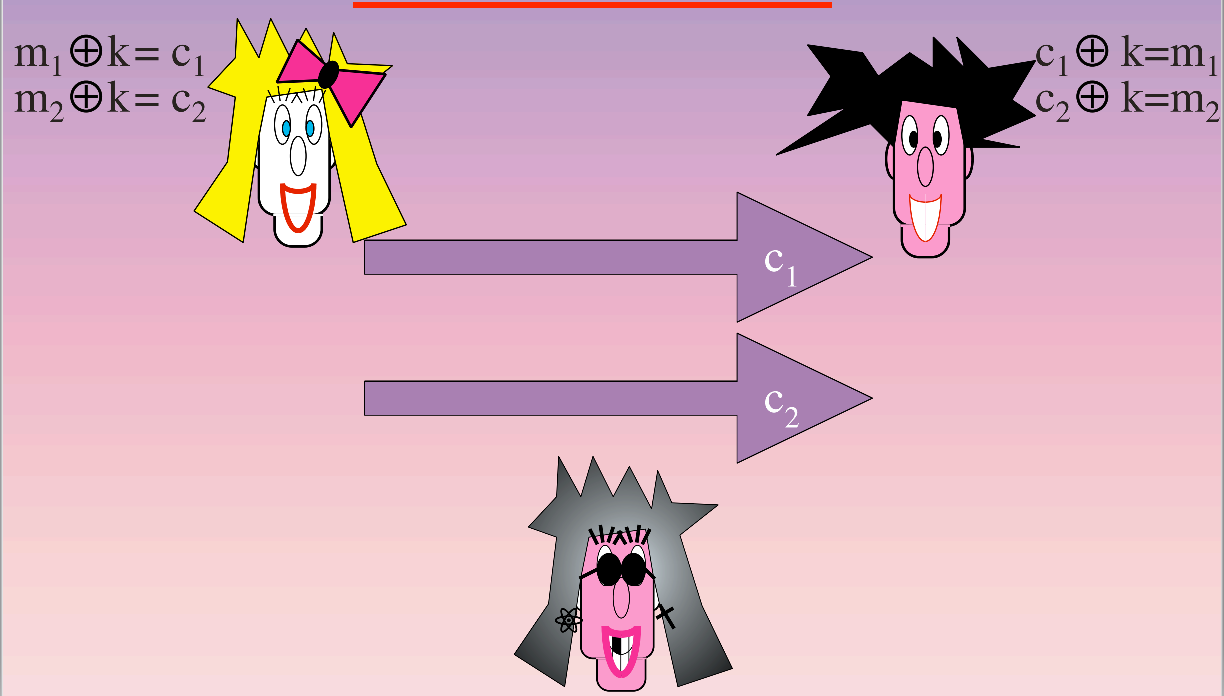
⊕ =

Information Theoretical Security

VISUAL DEMO

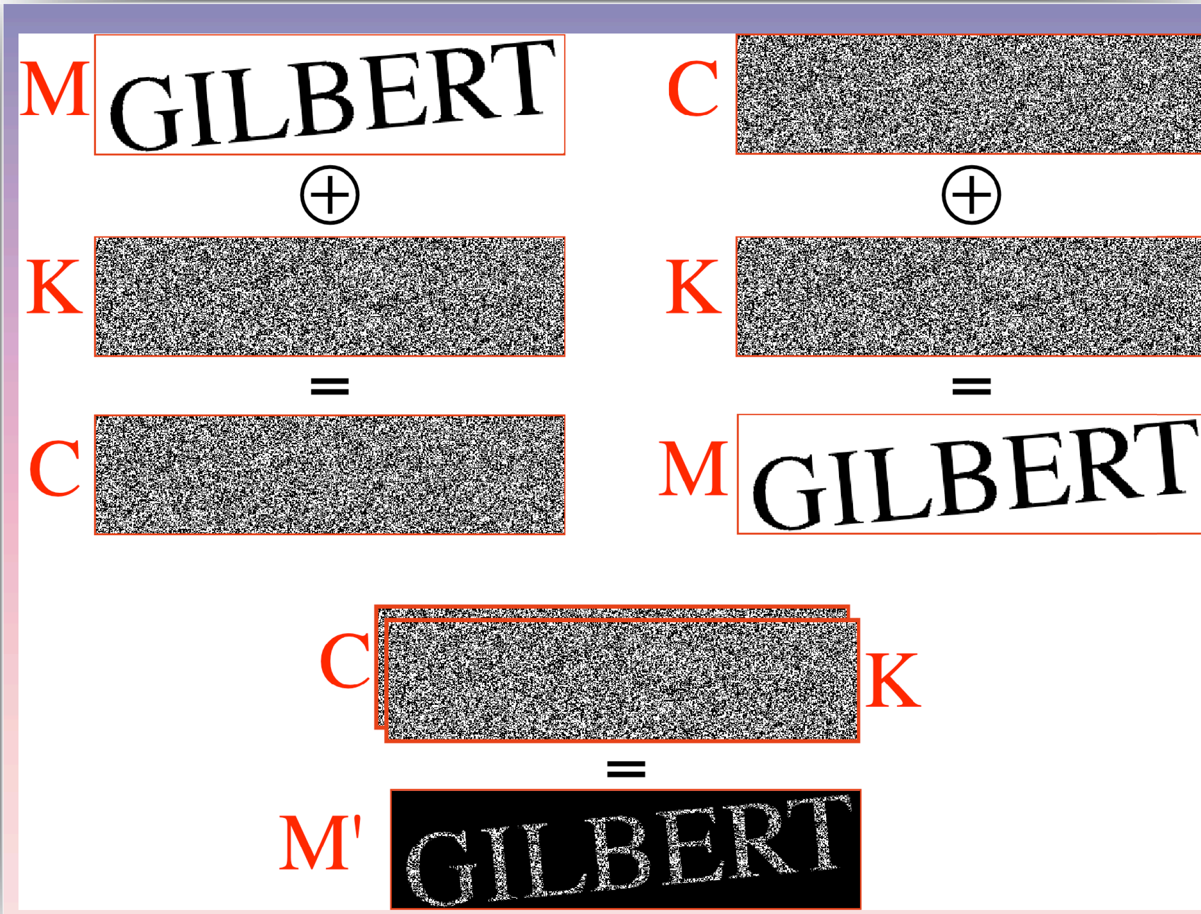


Vernam's One-Time-Pad

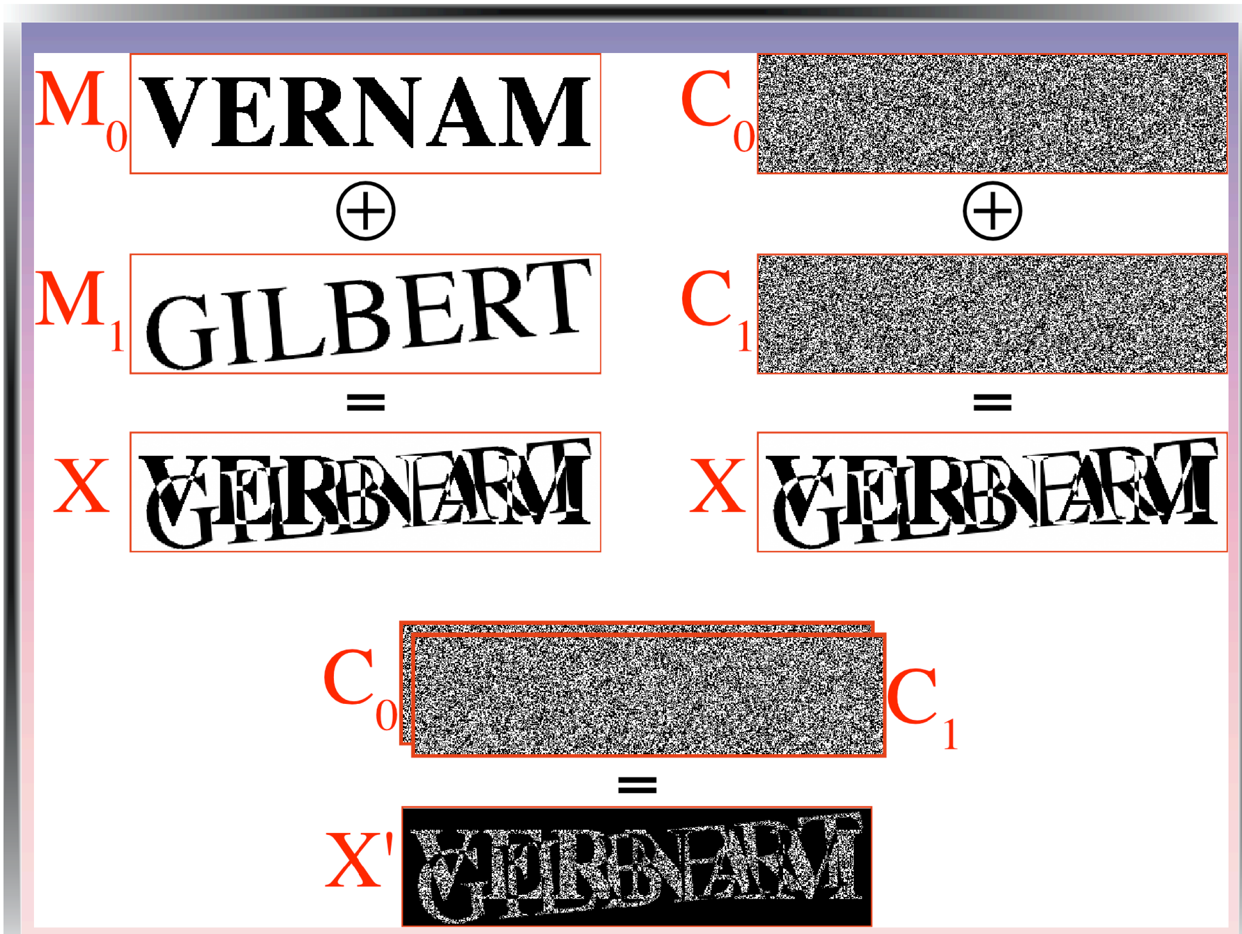


$c_1 \oplus c_2 = m_1 \oplus m_2$

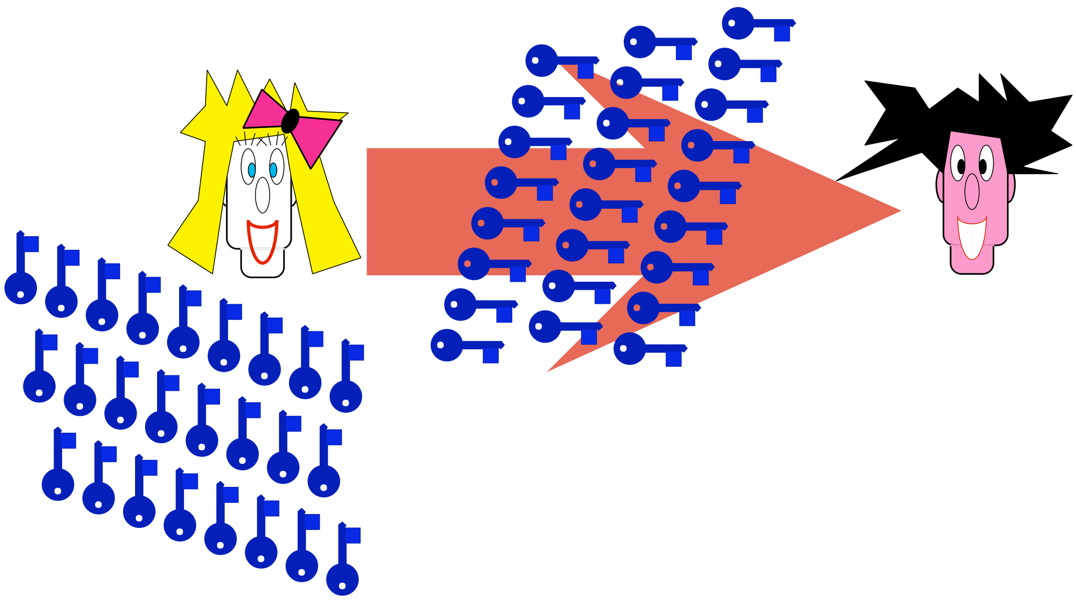
VISUAL DEMO



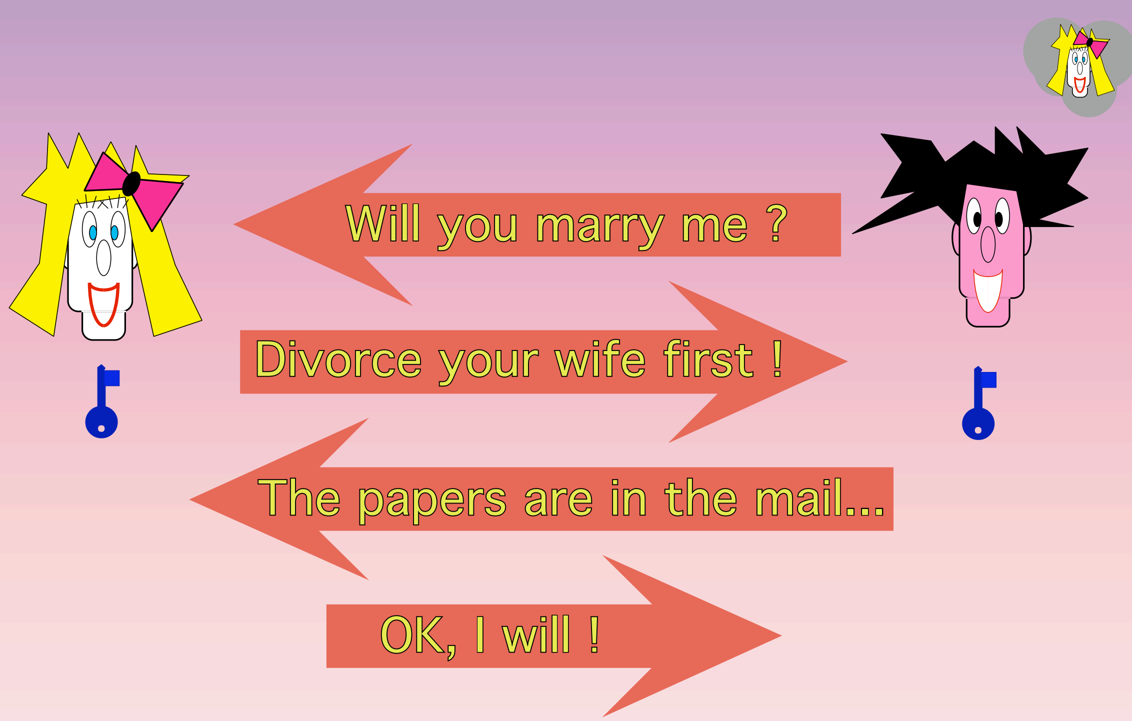
VISUAL DEMO

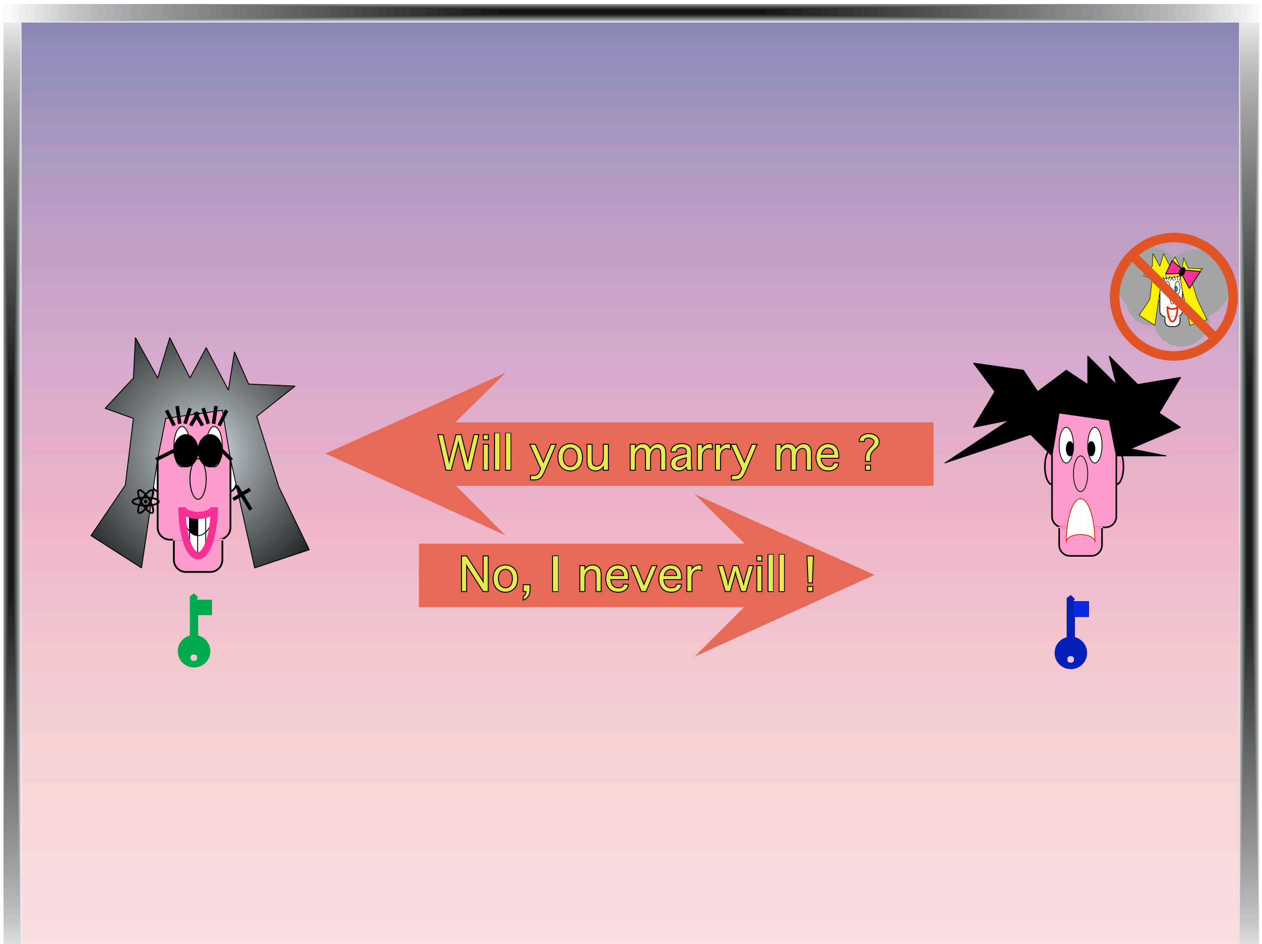


(1.1.1) key distribution PROBLEM

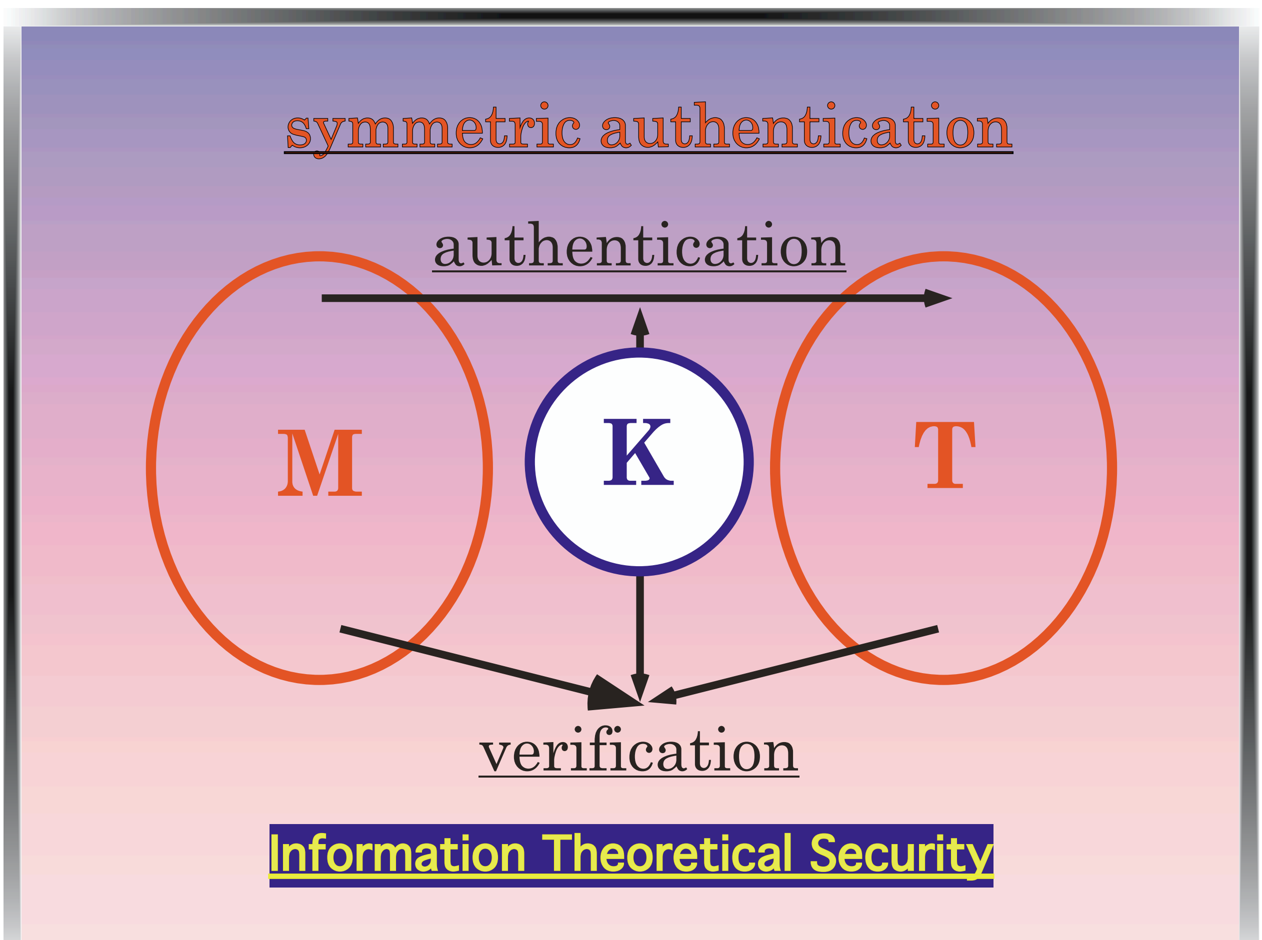


(1.1.3) Authentication



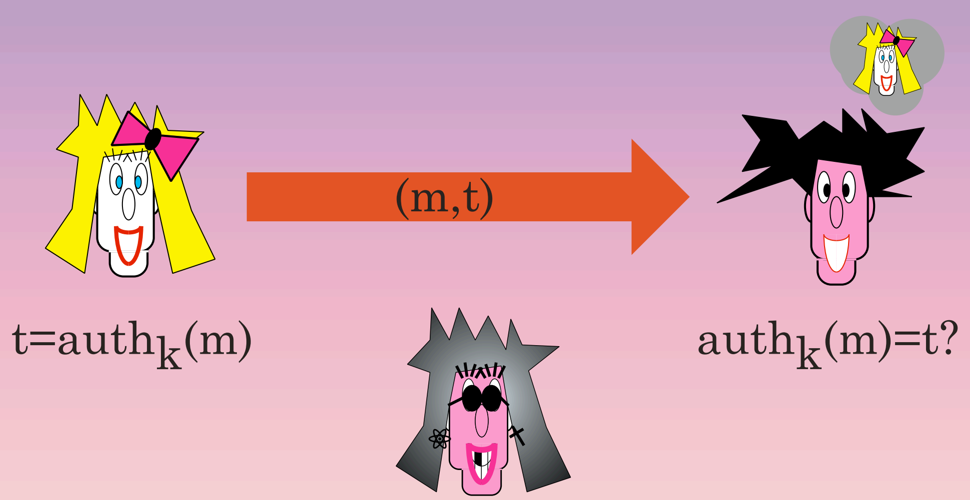


Will you marry me ?
No, I never will !



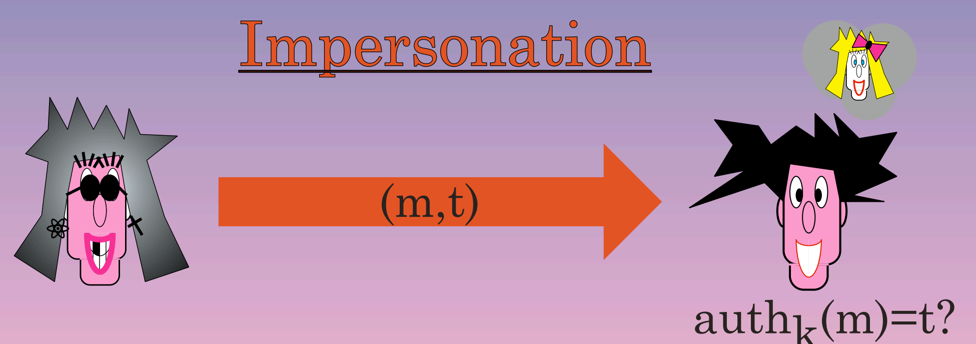
symmetric authentication
authentication
M
K
T
verification
Information Theoretical Security

Authentication

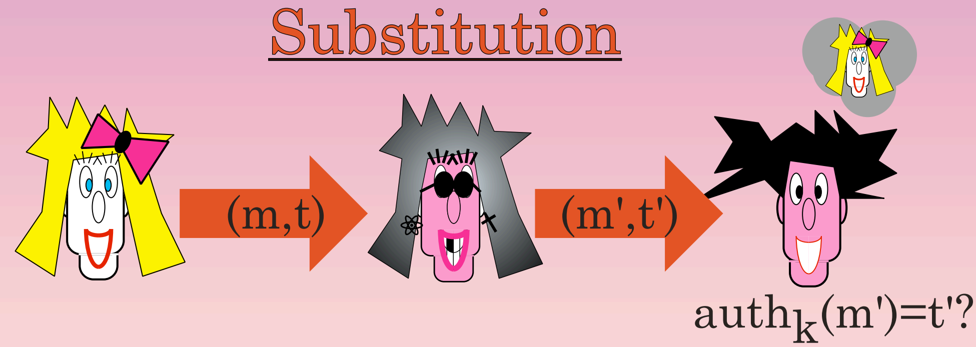


Information Theoretical Security

Impersonation



Substitution



Information Theoretical Security

WC One-Time-Authentication

$$auth_{\mathbf{M},b}(x) = \mathbf{M}x \oplus b$$

$|x| = n,\ |\mathbf{M}| = n \cdot n',\ |b| = n'$

$\forall m \in M, \forall t \in T$

$\Pr(auth_{\mathbf{M},b}(m)=t) = 1/|T| = 1/2^{n'}$

$\forall m \neq m' \in M, \forall t,t' \in T$

$\Pr\big(\ auth_{\mathbf{M},b}(m')=t' \mid auth_{\mathbf{M},b}(m)=t\ \big) = 1/|T| = 1/2^{n'}$

WC One-Time-Authentication and (linear) error correction

$$auth_{\mathbf{M},b}(x) = \mathbf{M}x \oplus b$$

$[\mathbf{I}:\mathbf{M}]m \oplus [0:b] = [m:t]$

$G=[\mathbf{I}:\mathbf{M}]$ (systematic) generating matrix of error correcting code

[0:b] error pattern = one-time pad encryption of tag

[m:t] systematic form of (message,tag)

(1.2) Complexity Theoretical Cryptography

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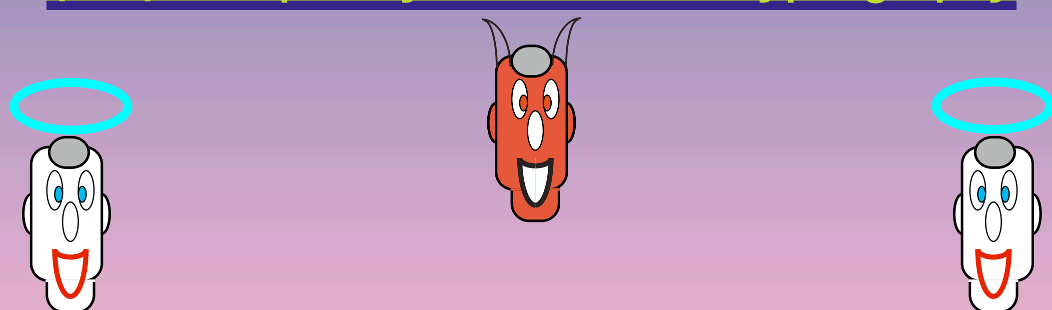
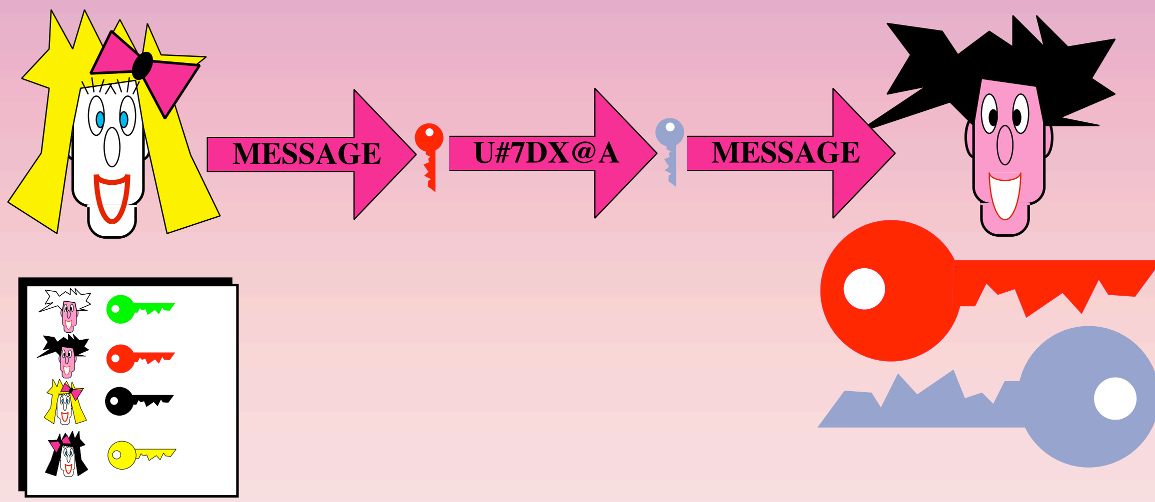


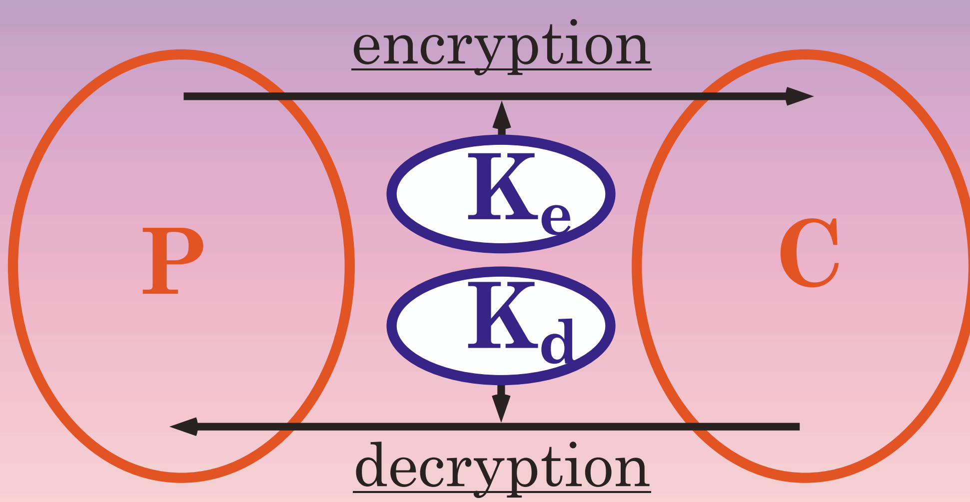
(1.2.1) Public key cryptosystem

(1.2.2) Digital signature scheme

(1.2.1) Public key cryptosystem



asymmetric encryption
(public-key cryptography)

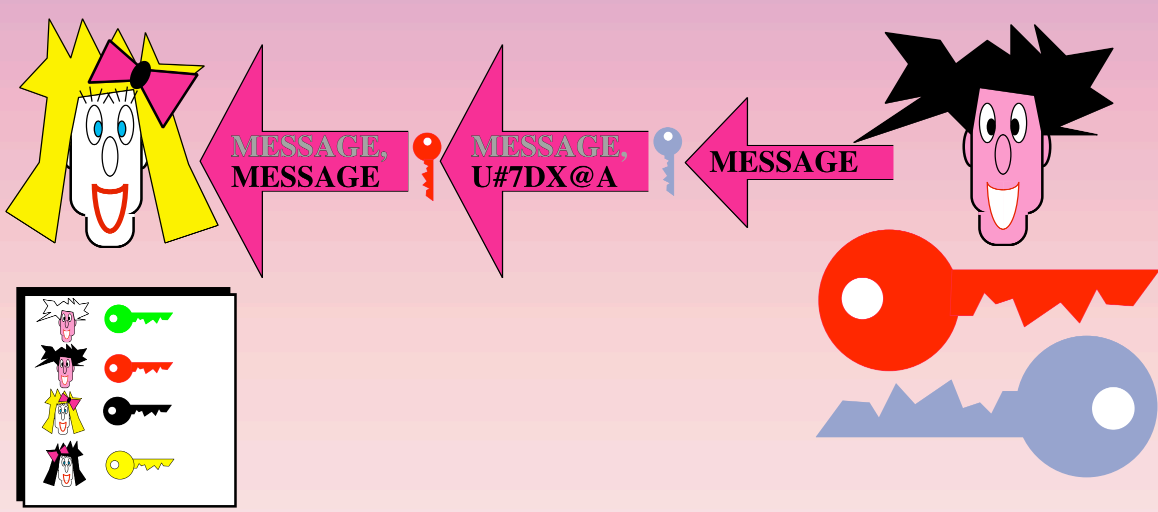


Complexity Theoretical Security

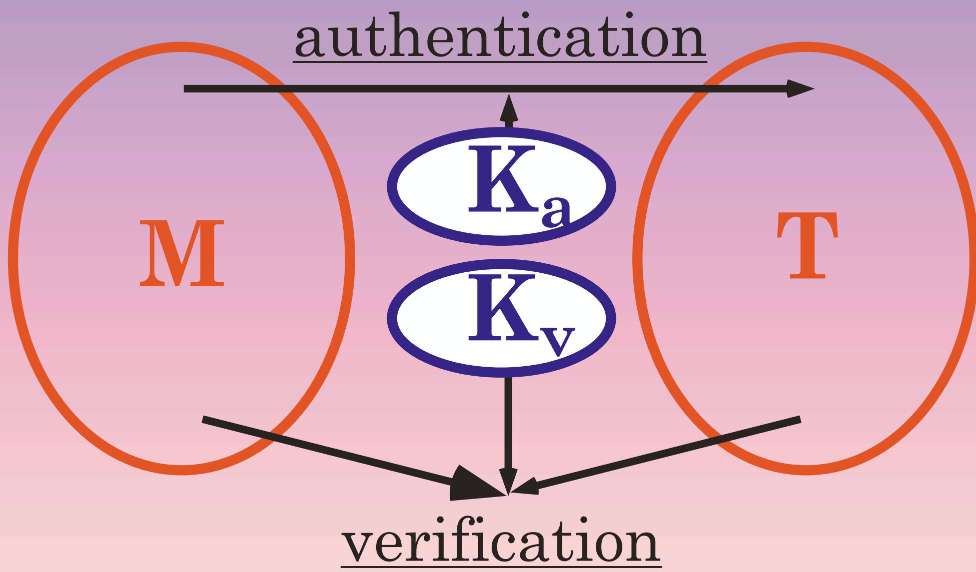
RSA public-key cryptosystem

- $n = p*q$, two large primes
- e s.t. $\gcd(e,(p-1)(q-1))=1$
- d s.t. $e*d = 1$ **mod** $(p-1)(q-1)$
- $K_e = (n,e)$, $K_d = (n,d)$
- **encryption** $E(m)$: m^e **mod** n
- **decryption** $D(c)$: c^d **mod** n

(1.2.2) Digital signature scheme



asymmetric authentication
(digital signature schemes)



Complexity Theoretical Security

RSA digital signature

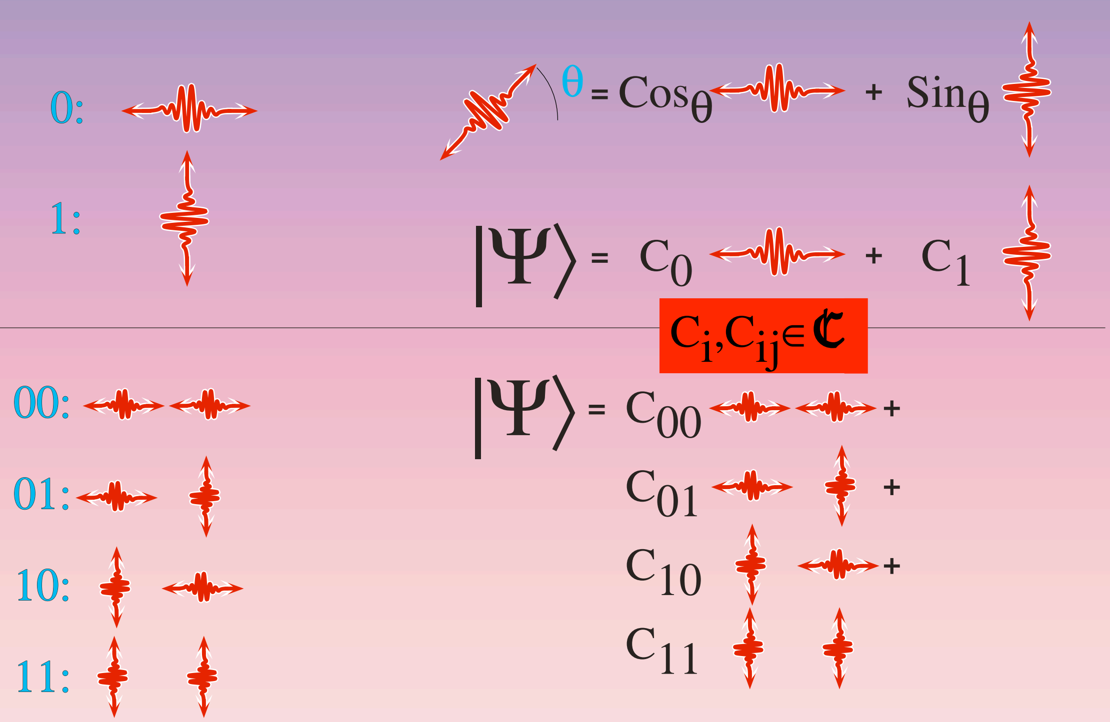
- n = p*q, two large primes
- e s.t. gcd(e,(p-1)(q-1))=1
- d s.t. e*d = 1 **mod** (p-1)(q-1)

- K_a = (n,d), K_v = (n,e)

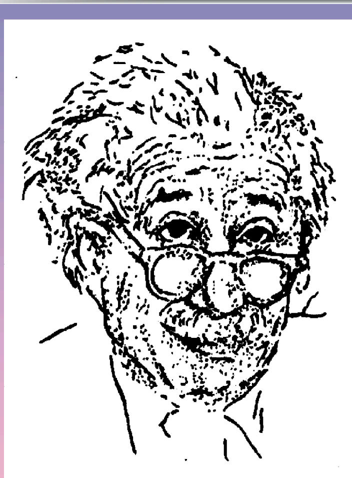
- **authentication** A(m): m^d **mod** n
- **verification** V(m,t): $t^e \equiv m$ **mod** n ?

(2) Quantum Information & Computations

Bits & QuBits

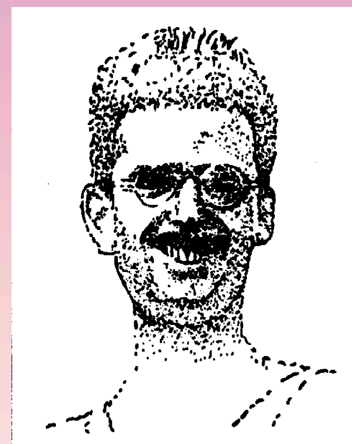


$$|¿?\rangle = {}^{1}/_{\sqrt{2}}\,|\mathbf{01}\rangle - {}^{1}/_{\sqrt{2}}\,|\mathbf{10}\rangle$$



Albert Einstein

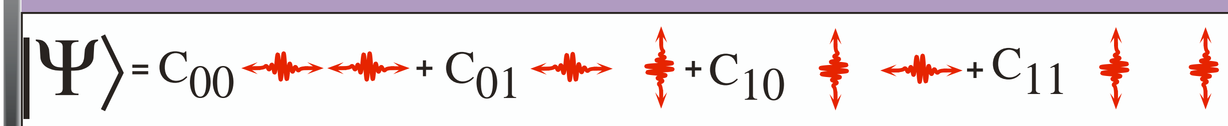
Boris Podolsky

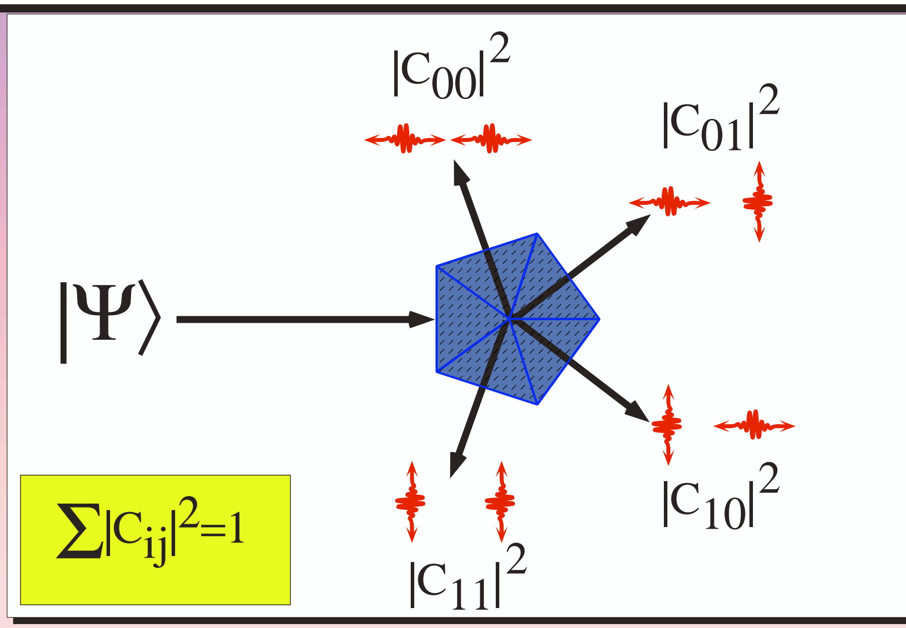


Nathan Rosen

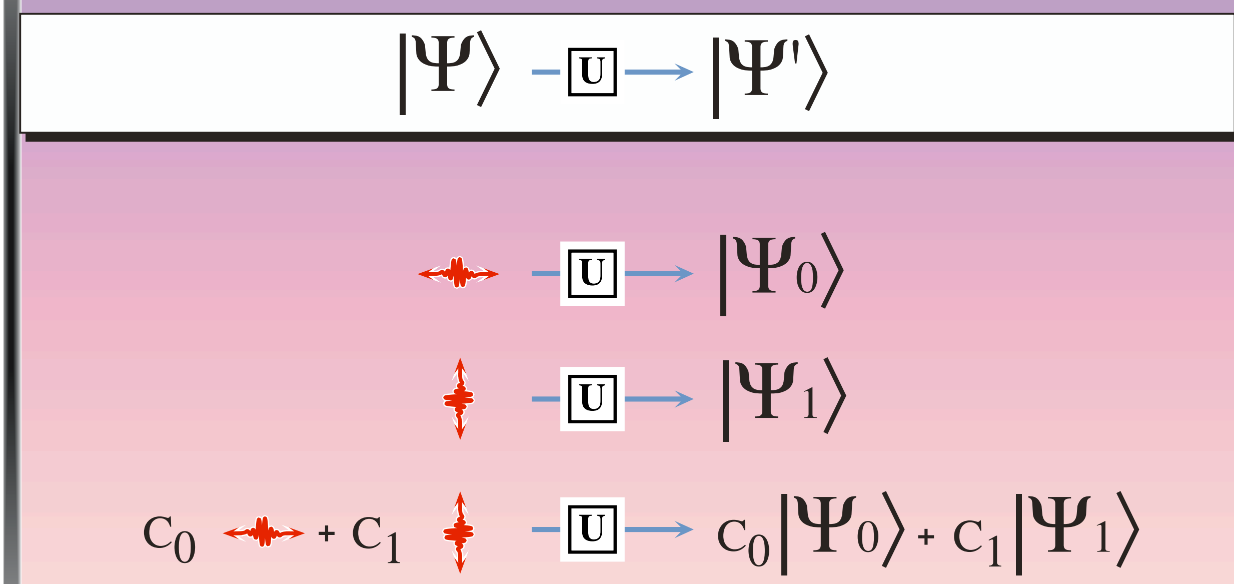
EPR

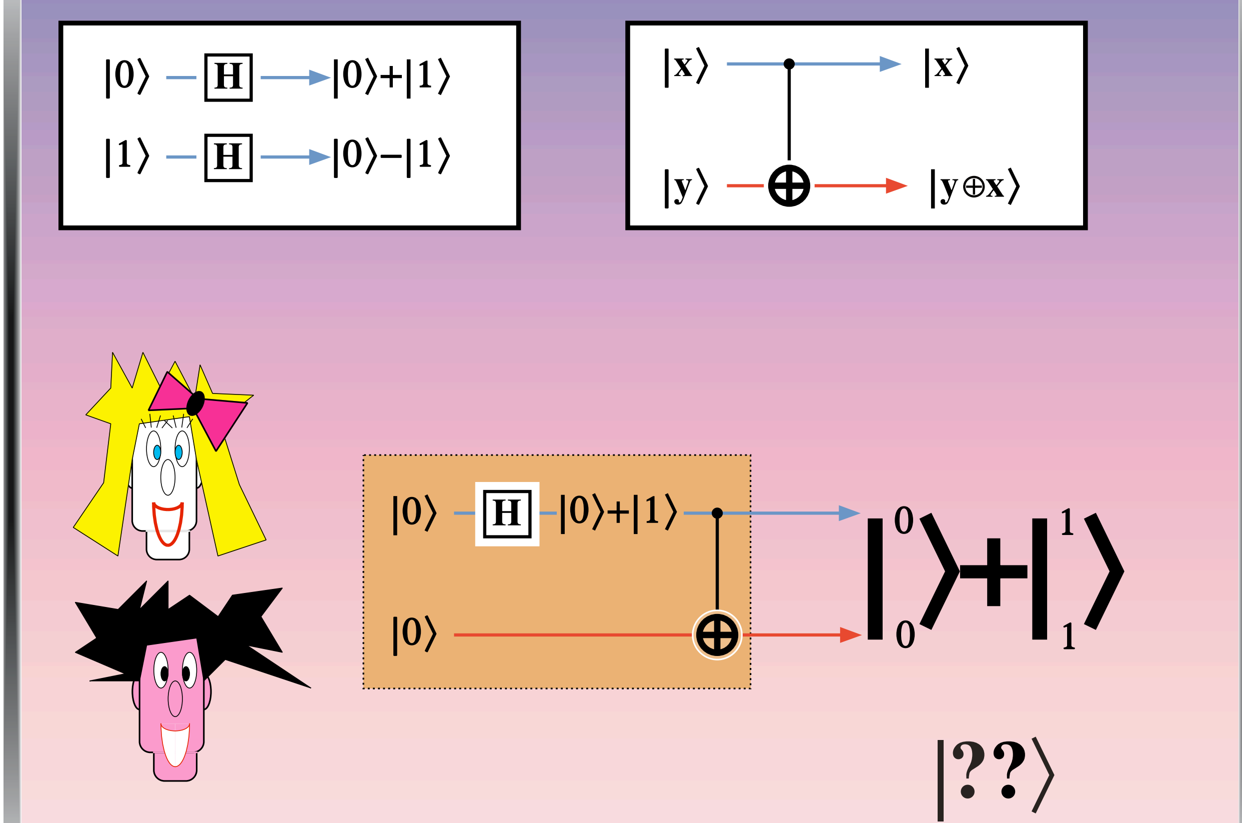
Quantum Measurements





Quantum Evolution: Unitary Operators





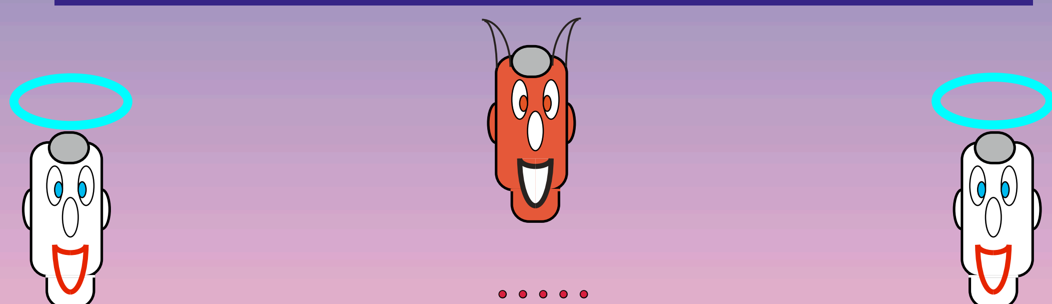
Classical & Quantum Information

00110111000110 Classical		Quantum
Copying:	Yes	NO
Measuring:	Yes	partial
Broadcasting:	Yes	NO
Superposing:	NO	Yes
Interfering:	NO	Yes

(3)
Quantum Cryptography

(3.1)
Information Theoretical Quantum Cryptography

(3.1) Information Theoretical Cryptography



(3.1.1) Key distribution : Q-key distribution +
Q-distillation (formerly purification)

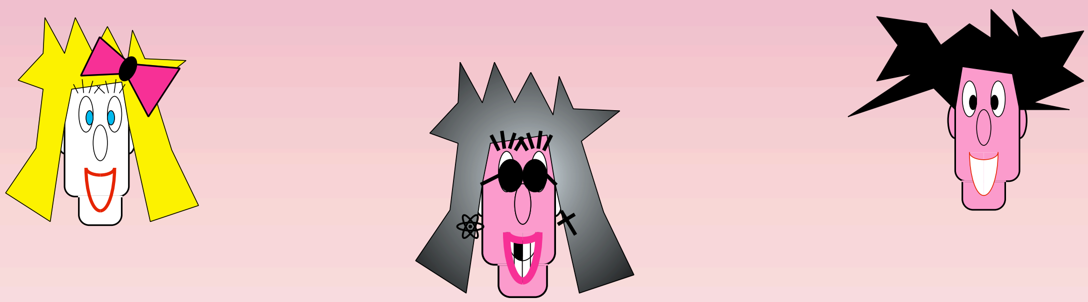
(3.1.2) One-time pad : one-time Q-pad (Q-teleportation)
Vernam Q-cipher

(3.1.3) one-time authentication : authenticated Q-teleportation +
one-time Q-authentication

(3.1.1) Key distribution

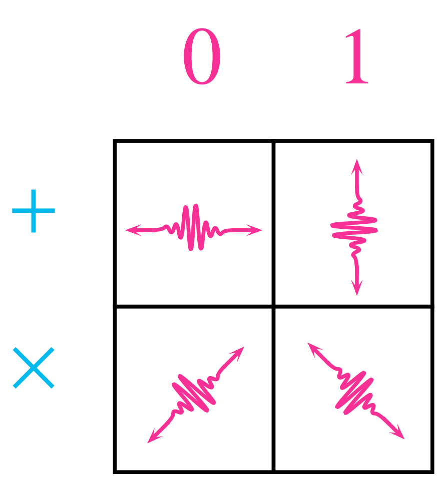
Classical key : Q-distribution of keys(BB84)
+ error-correction
+ privacy amplification

Quantum key : Q-key distribution(Ekert/Lo-Chau)
+ Q-error-correction (CSS) or
+ Q-Distillation (Purification)



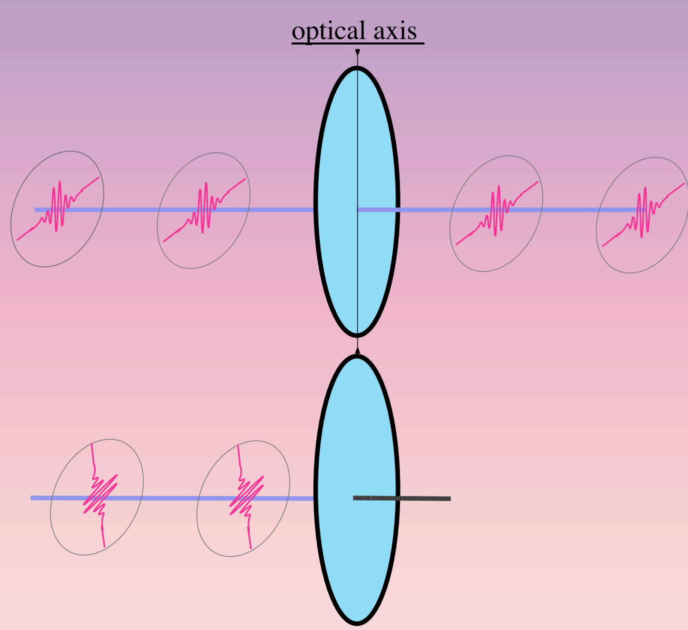
(3.1.1) Key distribution

Ambiguous Coding Scheme

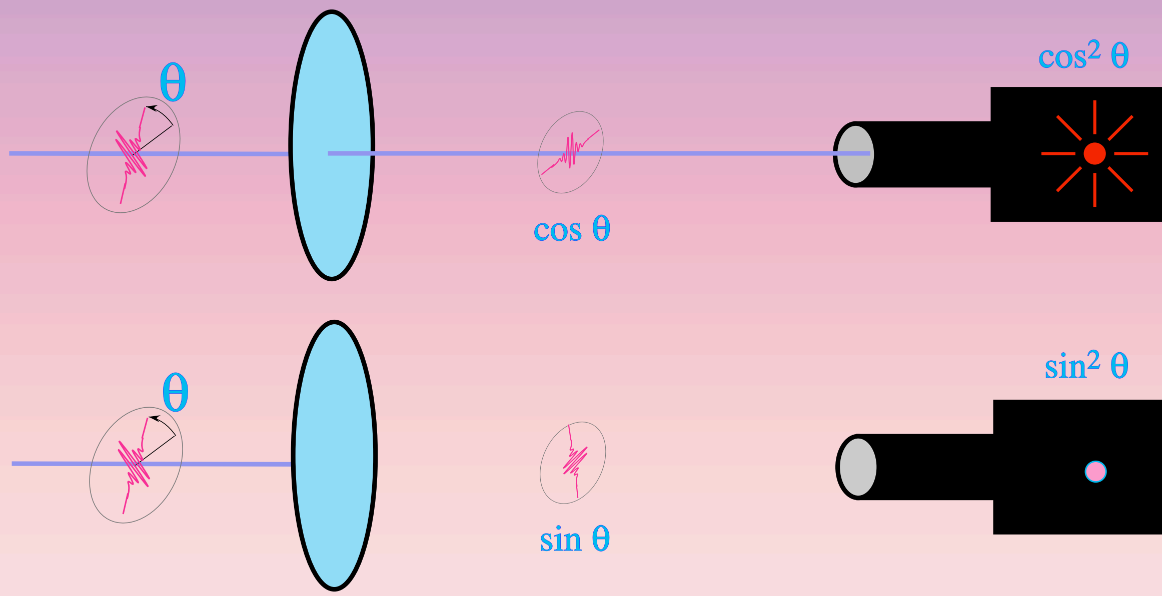


VISUAL DEMO

Polarizing Filter

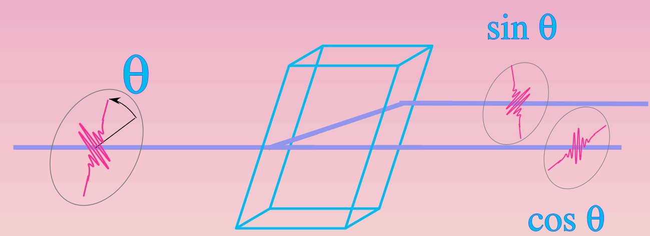


Polarizing Filter and photodetectors

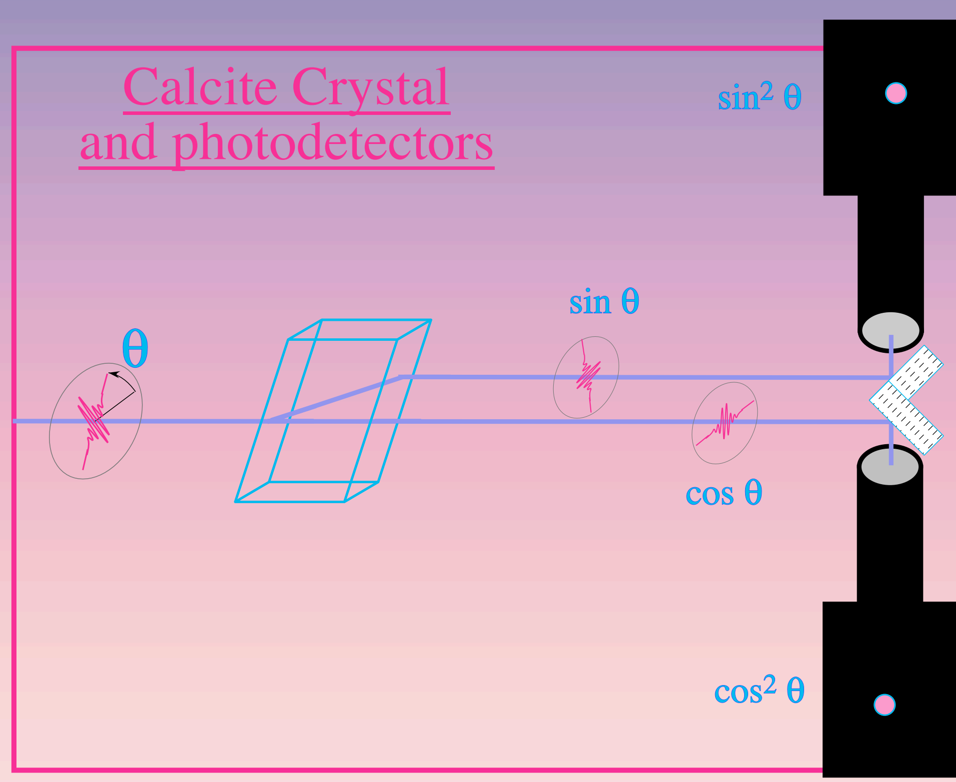


VISUAL DEMO

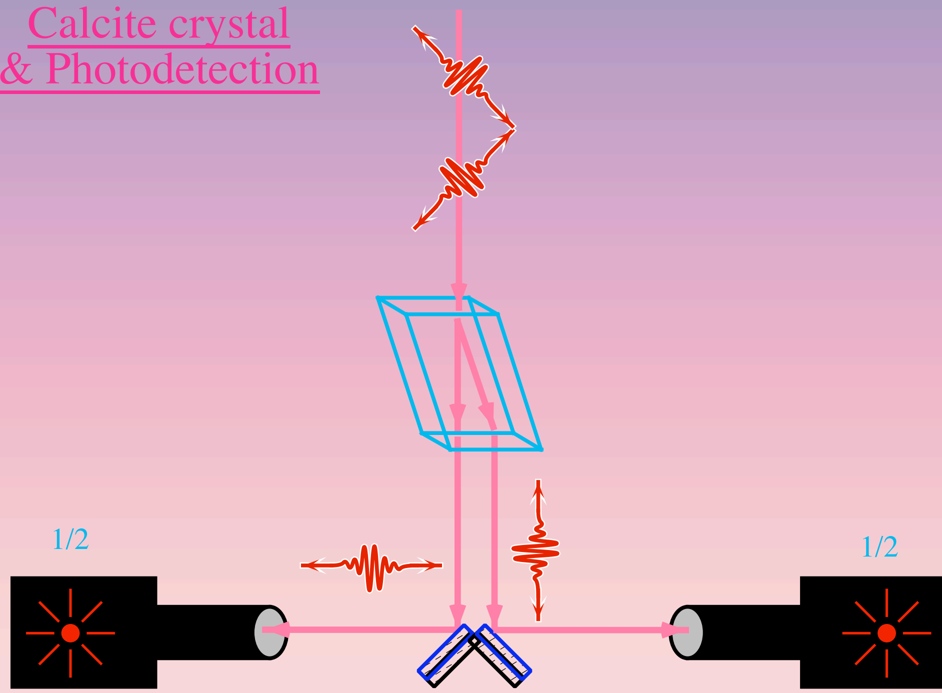
Calcite Crystal



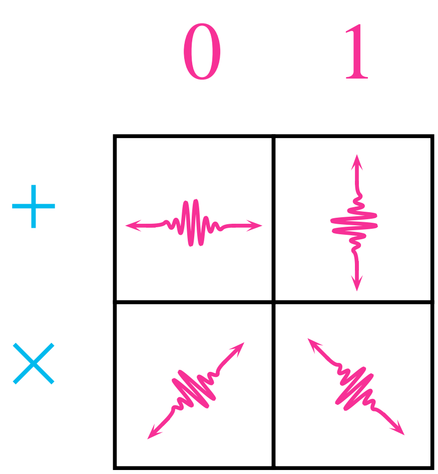
Calcite Crystal and photodetectors



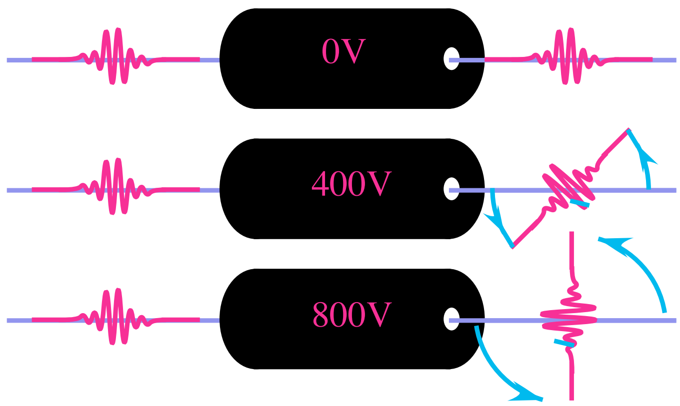
Calcite crystal & Photodetection

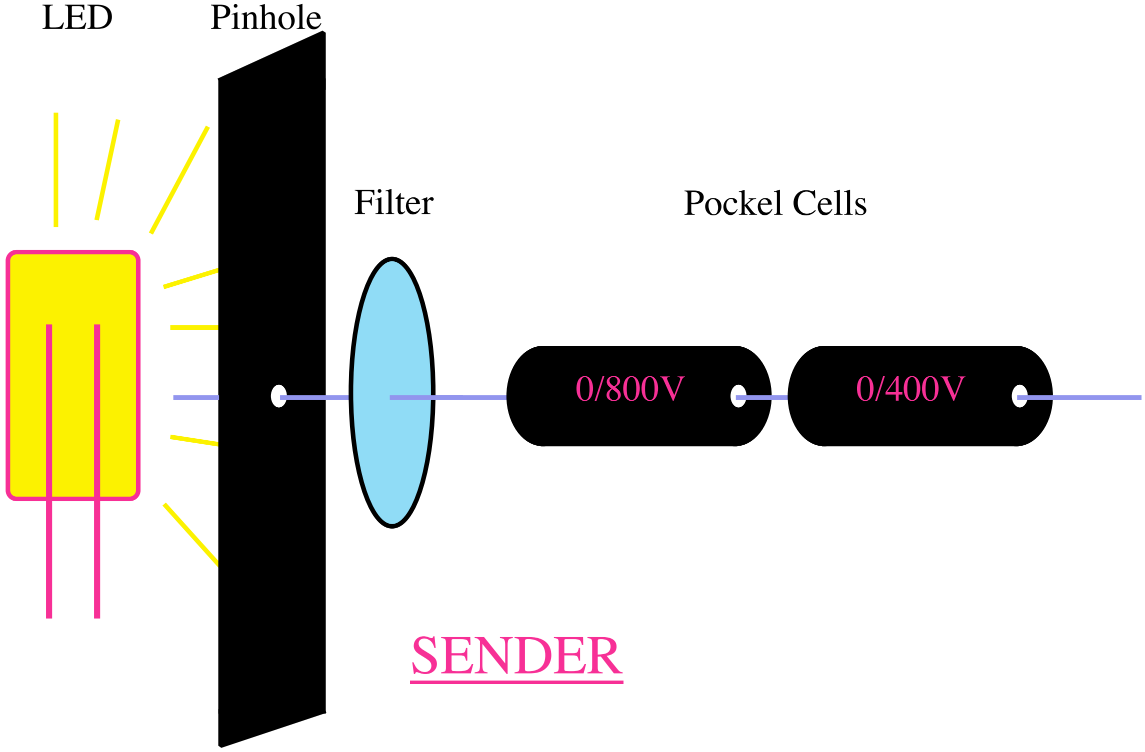


Ambiguous Coding Scheme

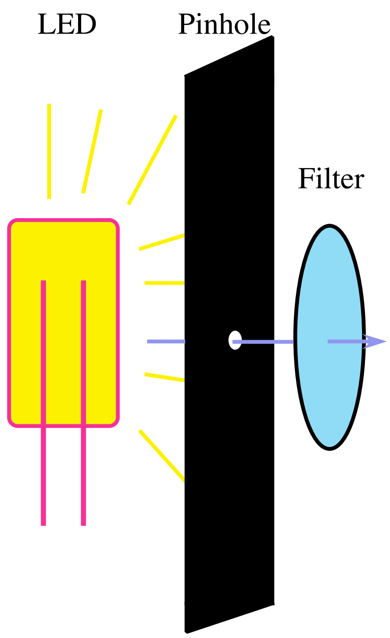


Pockel Cells





SENDER



Light source:

~ 1/10 photon per pulse

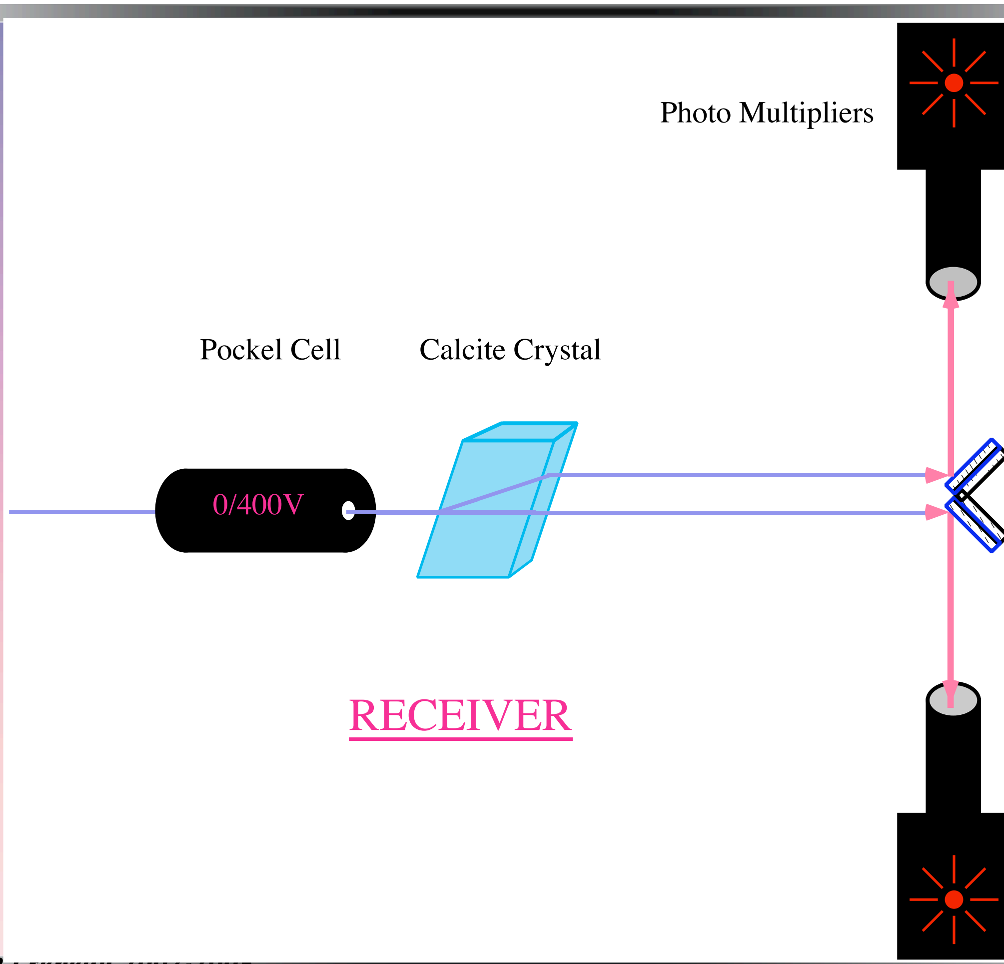
n = #photons per pulse follows a

Poisson distribution $Pr(n \leq x) = 1\text{-}e^{-x/10}$

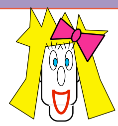
Problem:

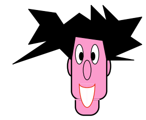
• may transmit multiple correlated

polarized photons



Q-distribution of keys



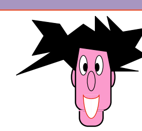


A:	0	1	1	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	0	0	0
	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	×	×	+	+	×	+	+	+	×	+	+	×	×	×	+	×	×	×	+	+	×	+	×	+
	0	0	1	0	0	1	0	0	1	0	0	0	0	1	1	1	0	0	0	1	1	0	0	0
A:	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	0	👎	👎	0	👎	1	👎	👎	1	👎	0	👎	👎	👎	👎	1	0	👎	👎	1	👎	0	0	0
B:	0			0		1			1		0					1	0			1		0	0	0
A:	0			0		1			1		0					1	1			1		0	0	0
A:	0					1					0						1							0
B:	=					=					=						≠							=
B:				0					1							1				1		0	0	
A:				0					1							1				1		0	0	

20%

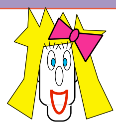
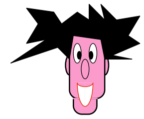
Bennett- Brassard

Q-distribution of keys



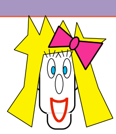
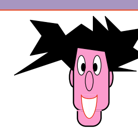
A:	0	1	1	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	0	0	0
	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+

Q-distribution of keys

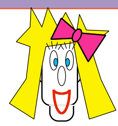
A:	0	1	1	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	0	0	0
	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	×	×	+	+	×	+	+	+	×	+	+	×	×	×	+	×	×	×	+	+	×	+	×	+
	0	0	1	0	0	1	0	0	1	0	0	0	0	1	1	1	0	0	0	1	1	0	0	0

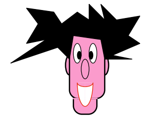
Q-distribution of keys

A:	0	1	1	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	0	0	0
	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	×	×	+	+	×	+	+	+	×	+	+	×	×	×	+	×	×	×	+	+	×	+	×	+
	0	0	1	0	0	1	0	0	1	0	0	0	0	1	1	1	0	0	0	1	1	0	0	0
A:	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+

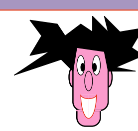
Q-distribution of keys





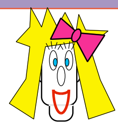
A:	0	1	1	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	0	0	0
	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	×	×	+	+	×	+	+	+	×	+	+	×	×	×	+	×	×	×	+	+	×	+	×	+
	0	0	1	0	0	1	0	0	1	0	0	0	0	1	1	1	0	0	0	1	1	0	0	0
A:	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	0	👎	👎	0	👎	1	👎	👎	1	👎	0	👎	👎	👎	👎	1	0	👎	👎	1	👎	0	0	0
B:	0			0		1			1		0					1	0			1		0	0	0
A:	0			0		1			1		0					1	1			1		0	0	0

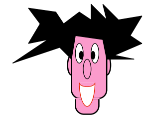
Q-distribution of keys



B:	0			0		1			1		0					1	0			1		0	0	0
A:	0			0		1			1		0					1	1			1		0	0	0

Q-distribution of keys

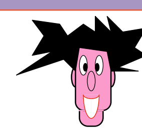




B:	0	0	1	1	0	1	0	1	0	0	0
A:	0	0	1	1	0	1	1	1	0	0	0
A:	0		1		0		1				0
B:	=		=		=		≠				=

20%

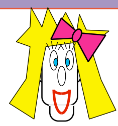
Q-distribution of keys

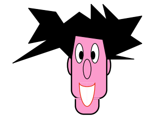


B:	=		=		=		≠				=
B:		0		1		1		1	0	0	
A:		0		1		1		1	0	0	

20%

Q-distribution of keys





B:	0	1	1	1	0	0
A:	0	1	1	1	0	0

20%

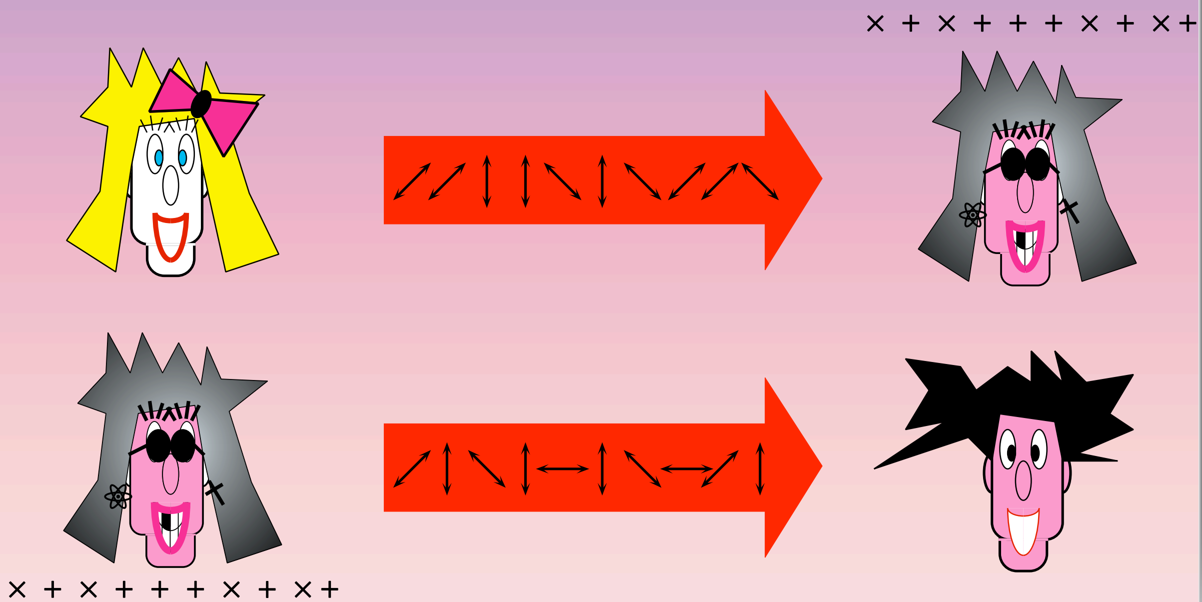
Q-distribution of keys

. . . . .

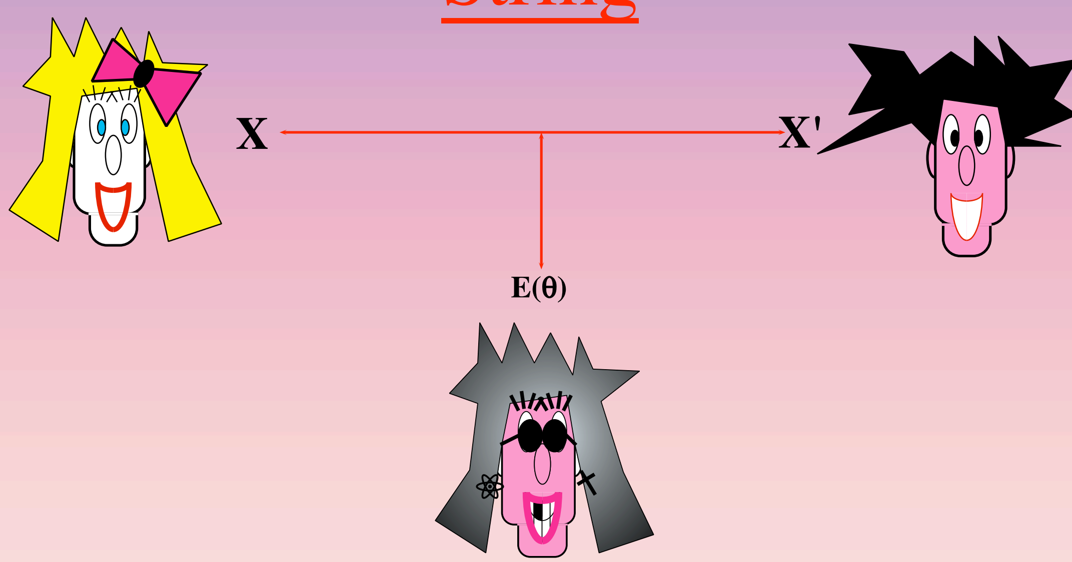
- Produces raw classical key
- Observed error rate indicates amount of eavesdropper information
- Error-correction is used to fix errors
- Random hash function is used to distill a smaller secret classical key

. . . . .

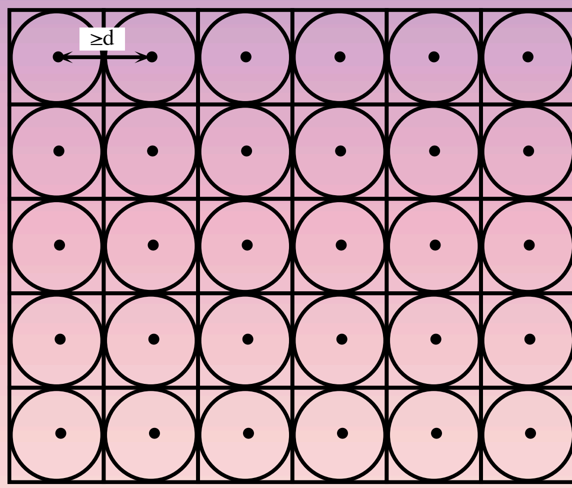
Information <--> Errors



Mostly Identical Partly Secret String



(classical) error-correcting codes



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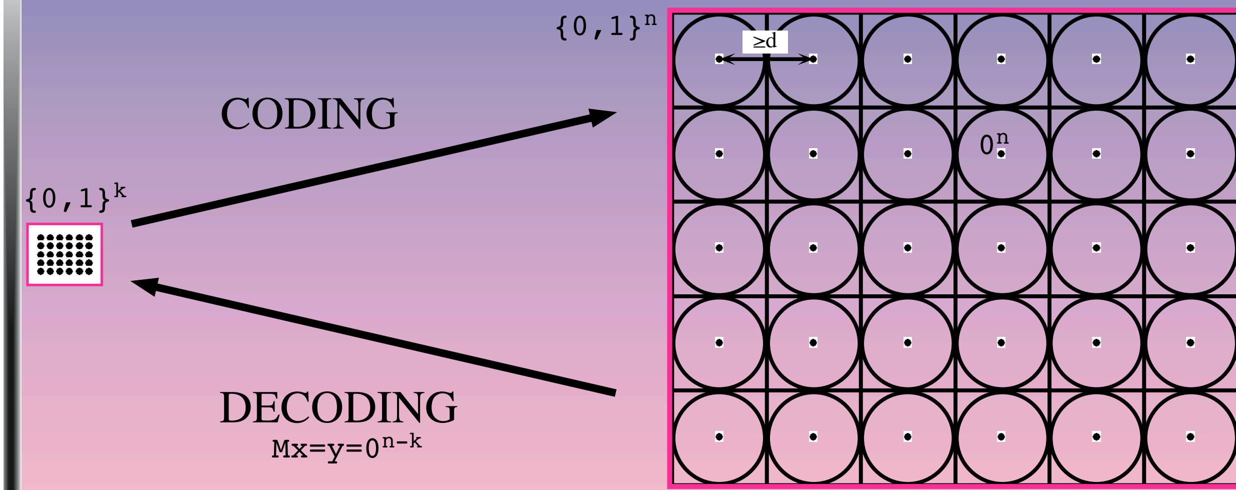


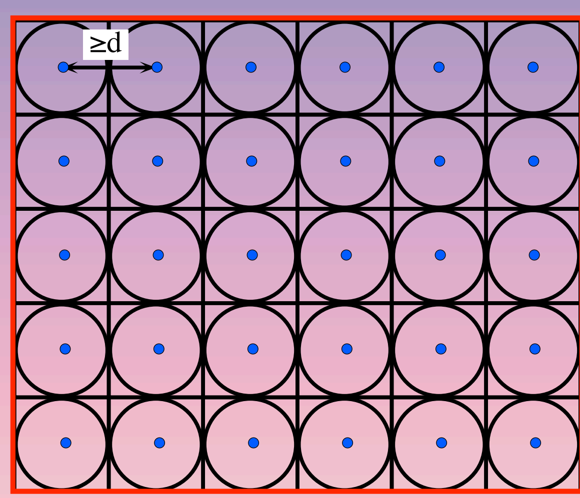
[n,k,d] linear code

$M \in \{0,1\}^{(n-k)\cdot n}$ is a Parity Check matrix

$C = \{ x \mid Mx = 0^{n-k} \}$

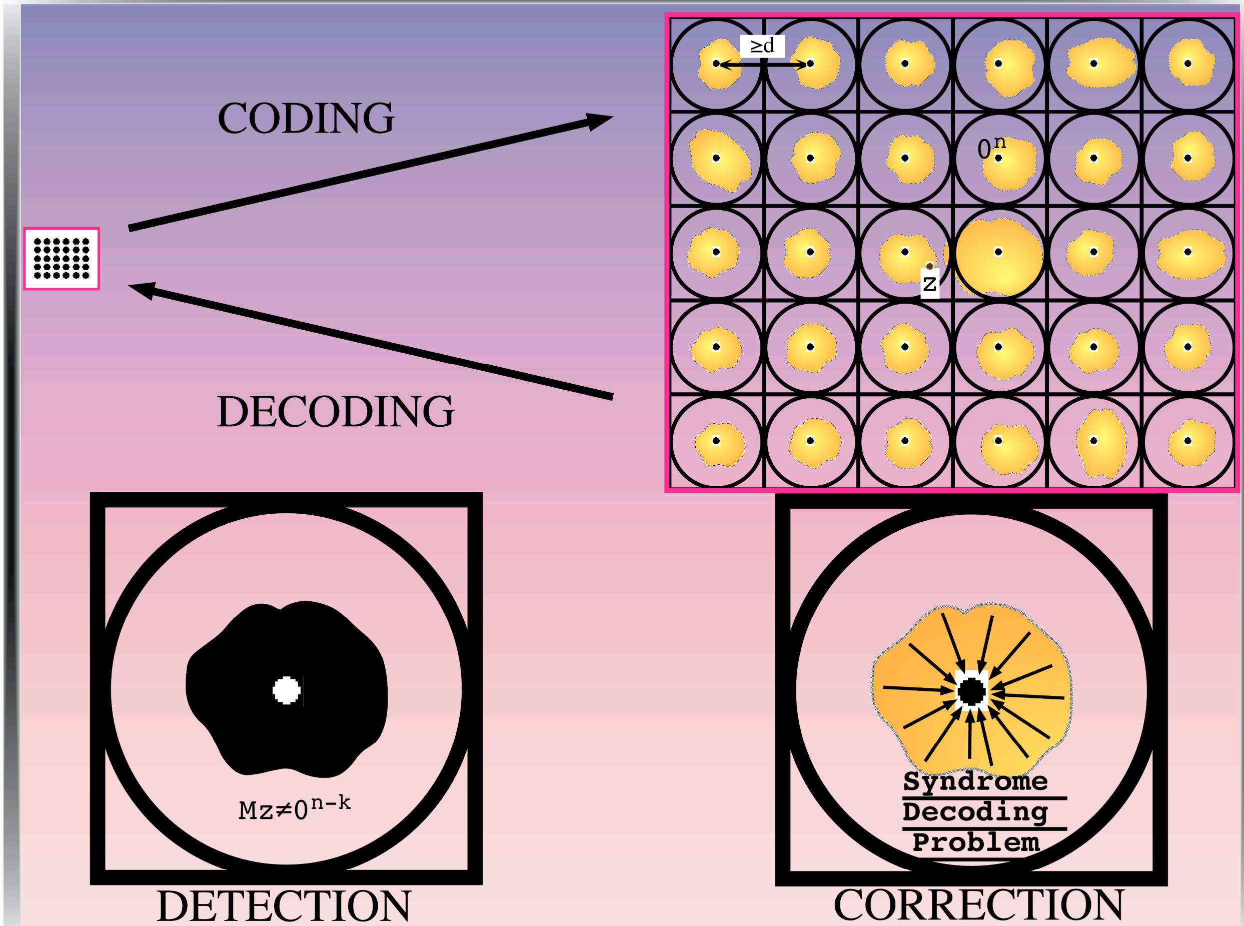
(classical) error-correcting codes

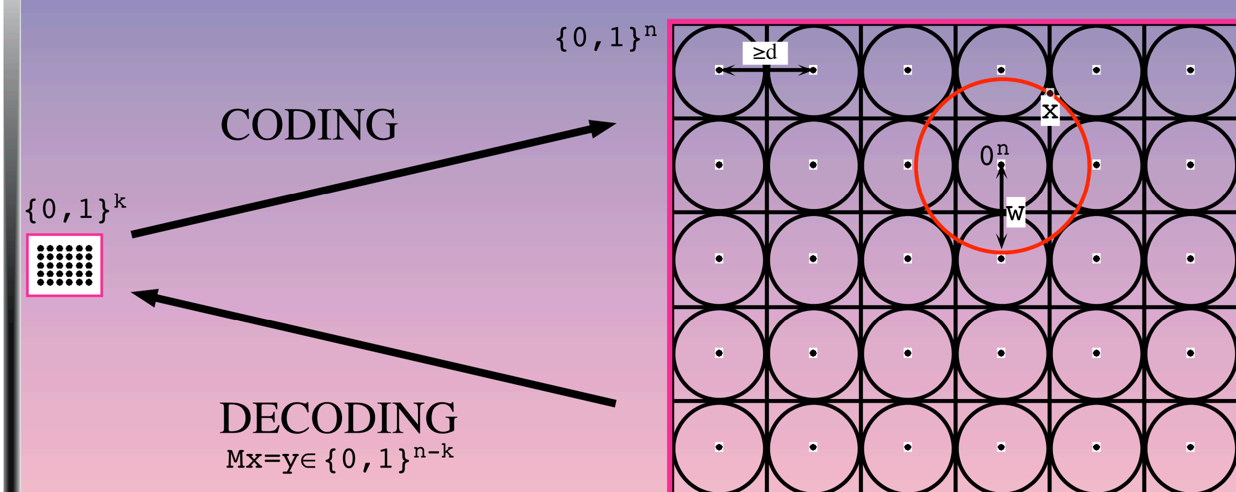
q^n (words)



q^k (codewords)

[n,k,d] linear error-correcting code
length n, dimension k,
corrects d-1 erasures, (d-1)/2 errors

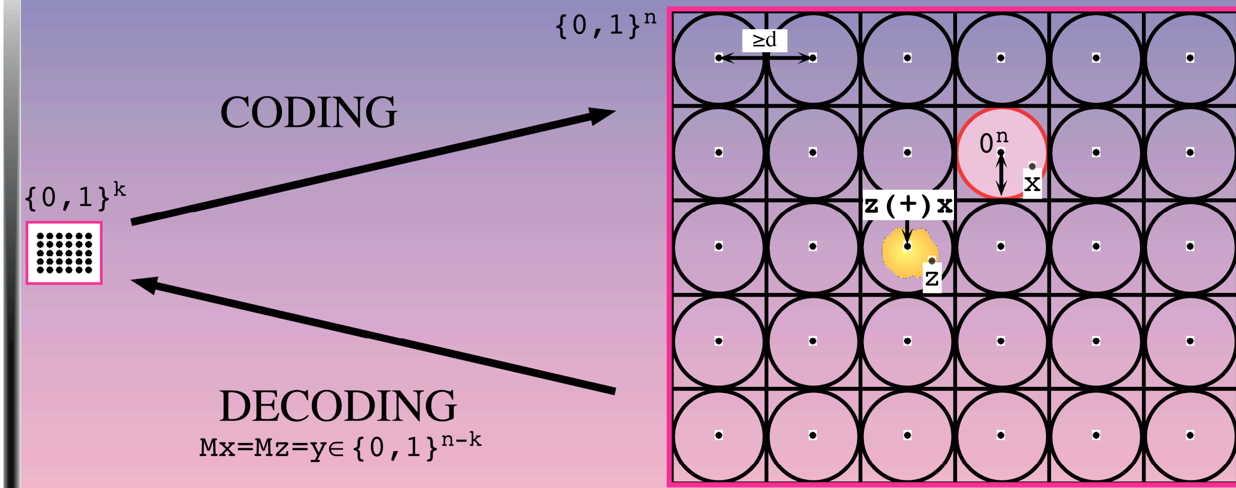




Syndrome Decoding Problem

Instance: PC matrix $M\in\{0,1\}^{(n-k)\cdot n}$, syndrome $y\in\{0,1\}^{n-k}$, weight $w\leq n$

Problem: is there a word $x\in\{0,1\}^n$, $|x|\leq w$ s.t. $Mx=y$?



CORRECTING(M,z) <= Syndrome Decoding Problem (M, w=(d-1)/2, y=Mz)

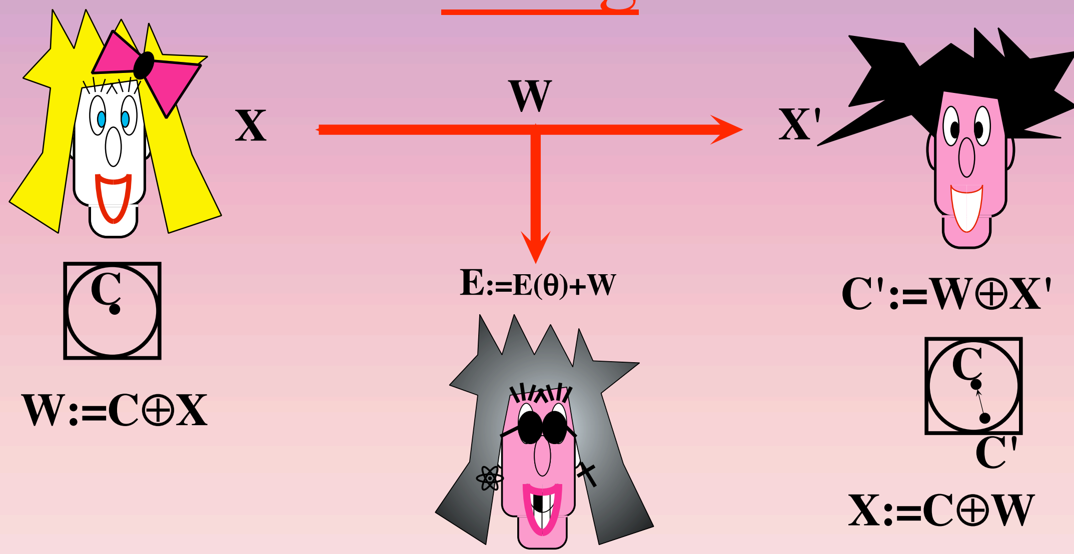
Instance: PC matrix $M \in \{0,1\}^{(n-k)\cdot n}$, $y=Mz \in \{0,1\}^{n-k}$, $w=(d-1)/2$

Problem: is there a word $x \in \{0,1\}^n$, $|x| \leq w$ s.t. $Mx=y$?

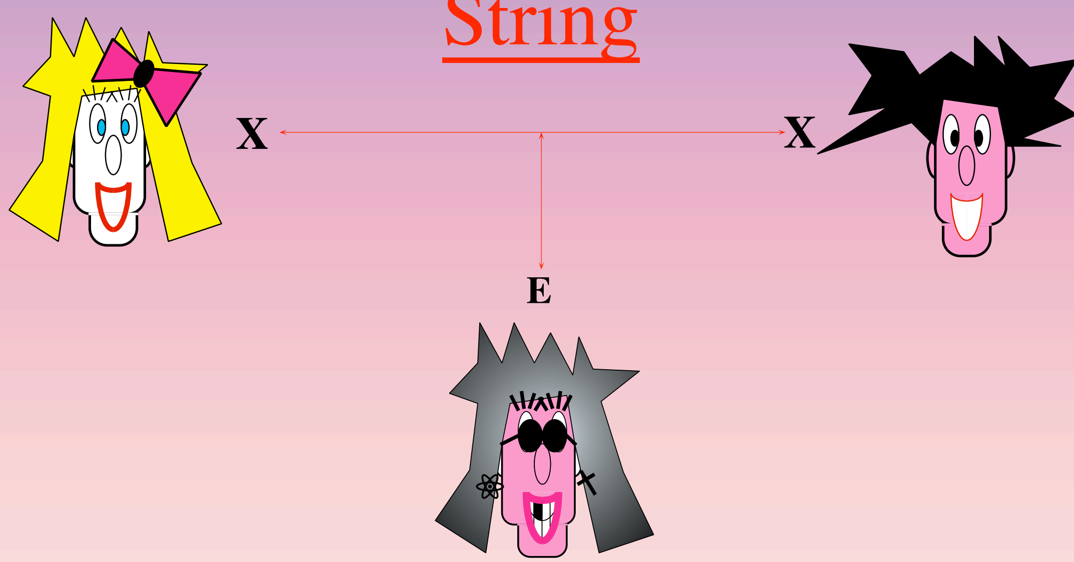
CORRECTING(M,z) = z(+)x

)

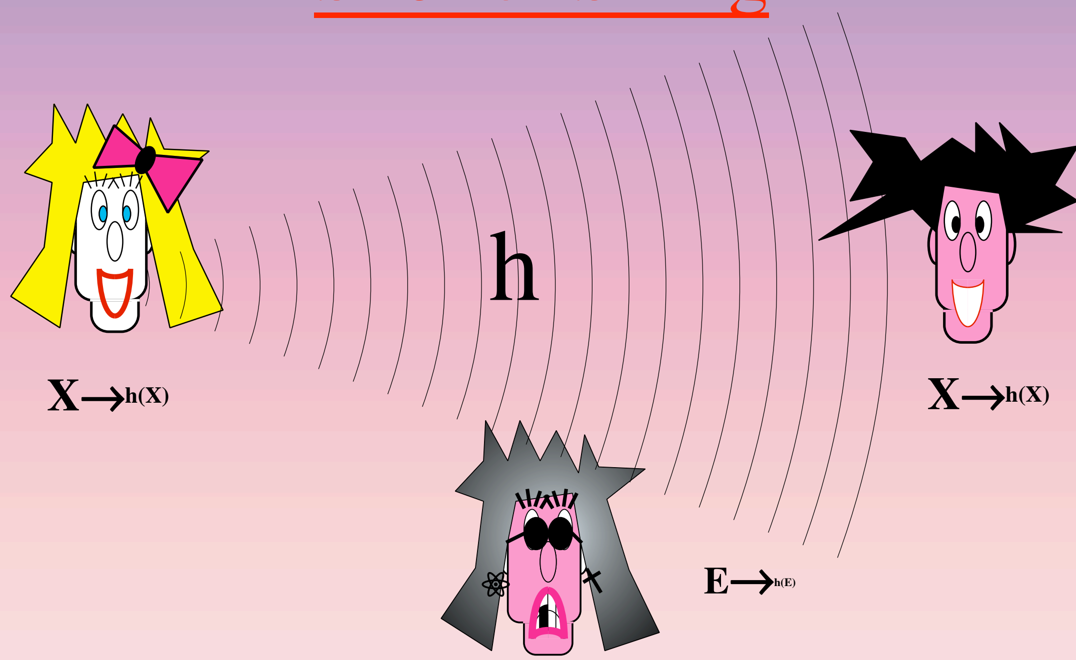
Identical Partly Secret String



Identical Partly Secret String

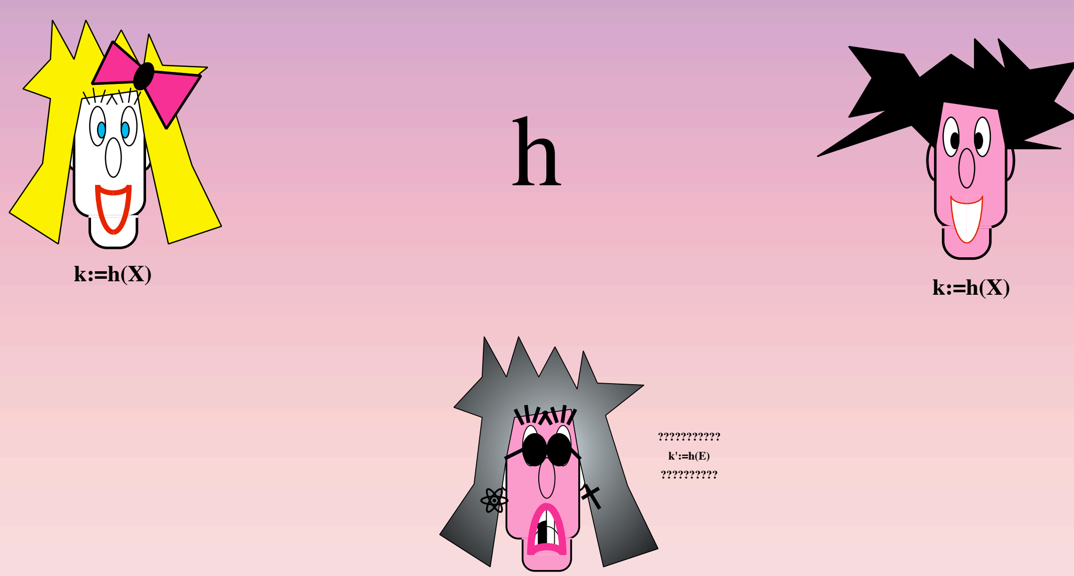


Identical Secret Shorter String

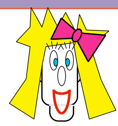


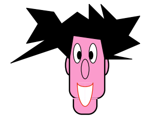
BBCM

$$H(\ h(X)\ |\ E,h\ ) > |h(X)| - 2^{(|h(X)|-H_\infty(X))}$$



Q-distribution of keys





A:	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	¿	¿	¿	¿	¿	¿	?	¿	¿	¿	¿	¿	¿	¿	¿	¿	?	¿	¿	¿	¿	¿	¿	¿
	×	×	+	+	×	+	+	+	×	+	+	×	×	×	+	×	×	×	+	+	×	+	×	+
	0	0	1	0	0	1	0	0	1	0	0	0	0	1	1	1	0	0	0	1	1	0	0	0
A:	×	+	×	+	+	+	×	×	×	×	+	+	+	+	×	×	×	+	×	+	+	+	×	+
B:	0	👎	👎	0	👎	1	👎	👎	1	👎	0	👎	👎	👎	👎	1	0	👎	👎	1	👎	0	0	0
A:	1			1		0			0		1					0	0			0		1	1	1
A:	1					0					1						0							1
B:	≠					≠					≠					≠	=							≠
B:				0					1							1				1		0	0	
A:				1					0							0				0		1	1	

20%

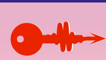
Ekert

(3.1.1) Key distribution

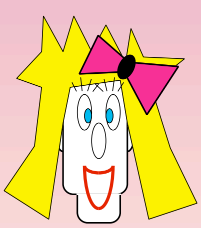
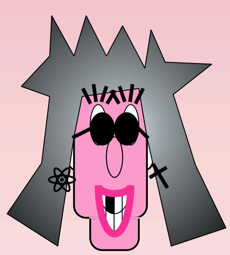
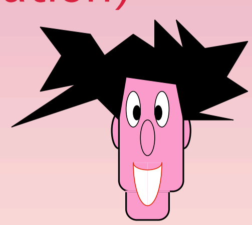
Classical key : Q-distribution of keys(BB84)

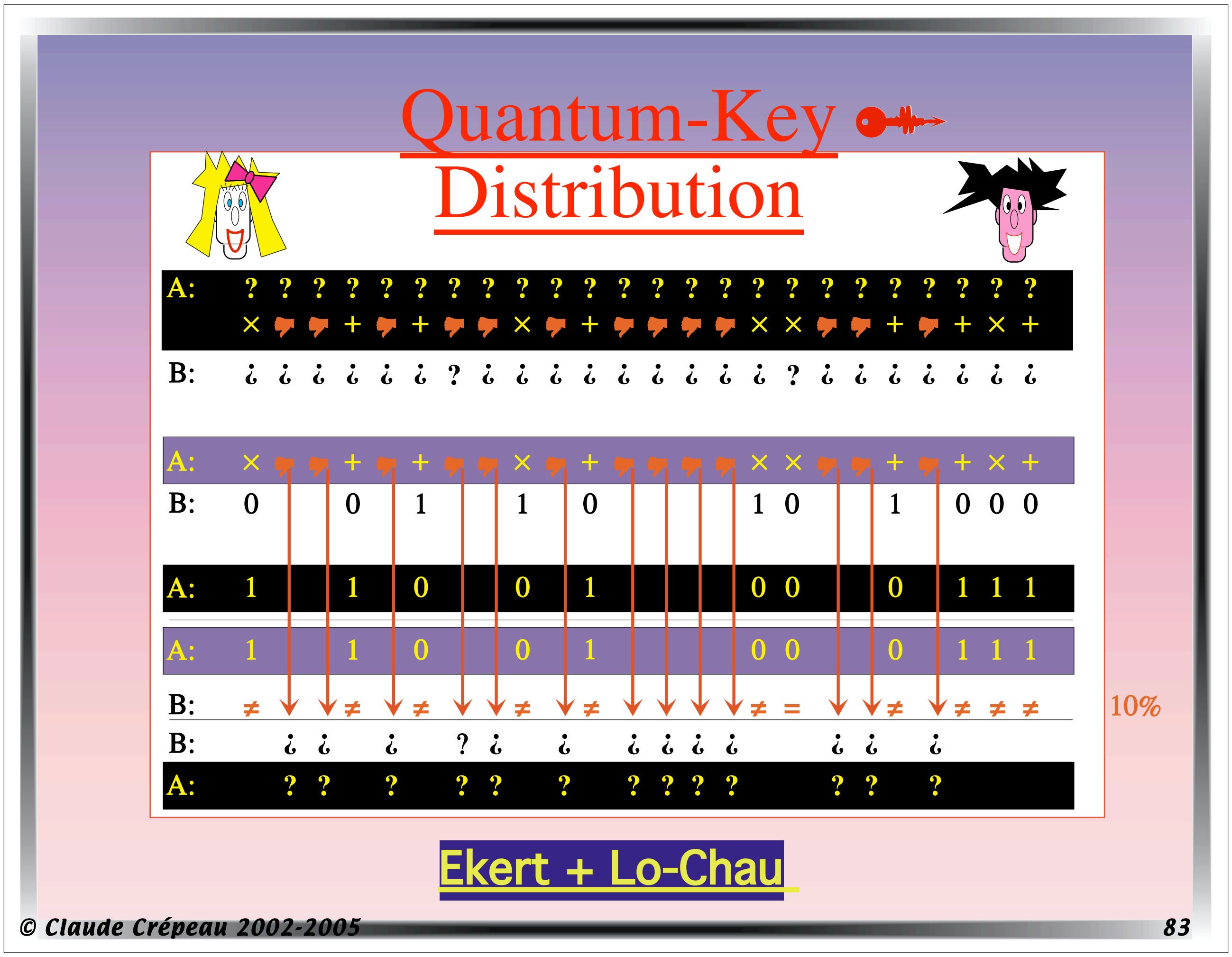
+ error-correction
+ privacy amplification

Quantum key : Q-key distribution(Ekert/Lo-Chau)



+ Q-error-correction (CSS) or
Q-Distillation (Purification)


Quantum-Key Distribution
A: ?
B: ¿ ¿ ¿ ¿ ¿ ¿ ? ¿ ¿ ¿ ¿ ¿ ¿ ¿ ¿ ¿ ? ¿ ¿ ¿ ¿ ¿ ¿ ¿
A:
B: 0 0 1 1 0 1 0 1 0 0 0
A: 1 1 0 0 1 0 0 0 1 1 1
A: 1 1 0 0 1 0 0 0 1 1 1
B: ≠ ≠ ≠ ≠ ≠ ≠ = ≠ ≠ ≠ ≠
B: ¿ ¿ ¿ ? ¿ ¿ ¿ ¿ ¿ ¿ ¿ ¿ ¿
A: ? ? ? ? ? ? ? ? ? ? ? ? ?
10%
Ekert + Lo-Chau
© Claude Crépeau 2002-2005
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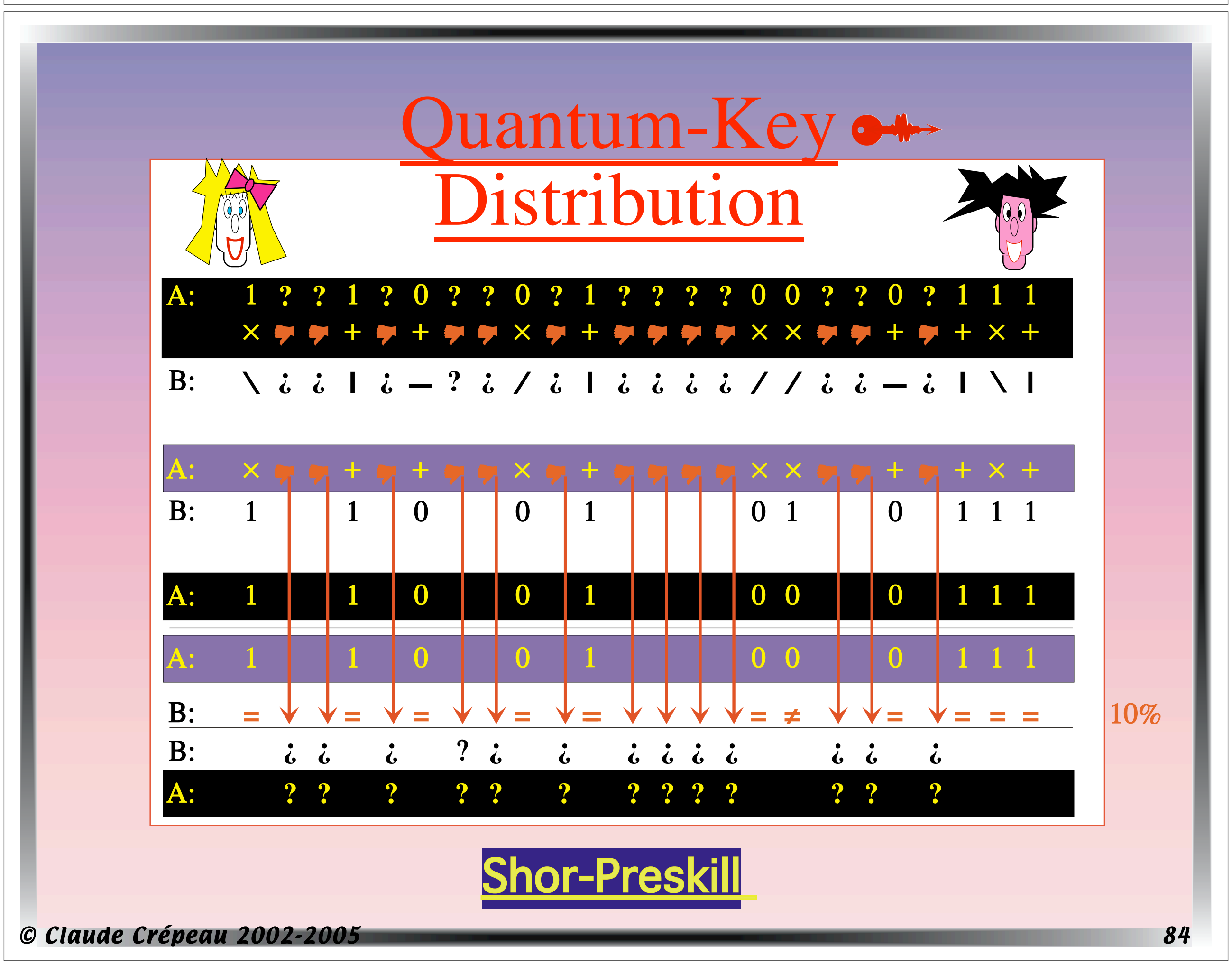
Quantum-Key Distribution
A: 1 ? ? 1 ? 0 ? ? 0 ? 1 ? ? ? ? 0 0 ? ? 0 ? 1 1 1
B: \ ¿ ¿ | ¿ — ? ¿ / ¿ | ¿ ¿ ¿ ¿ / / ¿ ¿ — ¿ | \ |
A:
B: 1 1 0 0 1 0 1 0 1 1 1
A: 1 1 0 0 1 0 0 0 1 1 1
A: 1 1 0 0 1 0 0 0 1 1 1
B: = = = = = = ≠ = = = =
B: ¿ ¿ ¿ ? ¿ ¿ ¿ ¿ ¿ ¿ ¿ ¿ ¿
A: ? ? ? ? ? ? ? ? ? ? ? ? ?
10%
Shor-Preskill
© Claude Crépeau 2002-2005
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Quantum-Key Distribution

.....

- Produces raw quantum key (EPR states)

- Observed error rate indicates amount of impurity of EPR states

- Quantum error-correction (CSS) is used to purify raw EPR states into a smaller pure set

.....

(

Q: (over GF(3))

$$|0\rangle \rightarrow |000\rangle + |111\rangle + |222\rangle$$
$$|1\rangle \rightarrow |012\rangle + |120\rangle + |201\rangle$$
$$|2\rangle \rightarrow |021\rangle + |102\rangle + |210\rangle$$

$$Q|\psi\rangle = H_1 \otimes H_2 \otimes H_3$$

Q=[[3,1,2]] corrects 2-1=1 erasure.

$$|0\rangle \otimes H_2 \otimes H_3 \rightarrow (-H_2 - H_3 \text{ mod } 3) \otimes H_2 \otimes H_3$$
$$H_1 \otimes |0\rangle \otimes H_3 \rightarrow H_1 \otimes (-H_3 - H_1 \text{ mod } 3) \otimes H_3$$
$$H_1 \otimes H_2 \otimes |0\rangle \rightarrow H_1 \otimes H_2 \otimes (-H_1 - H_2 \text{ mod } 3)$$

Calderbank-Shor-Steane Q-ECCs

Let C_1, C_2 be two linear codes such that

$$\{0\} \subseteq C_2 \subseteq C_1 \subseteq F^n \qquad \{0\} \subseteq C_1^{\perp} \subseteq C_2^{\perp} \subseteq F^n$$

For $v \in C_1$ define

$$v \rightarrow \frac{1}{\sqrt{|C_2|}} \sum_{w \in C_2} |v+w\rangle$$

For $v \in C_2^{\perp}$ define

$$v \rightarrow \frac{1}{\sqrt{|C_1^{\perp}|}} \sum_{w \in C_1^{\perp}} |v+w\rangle$$

$$Q = \left\{ \frac{1}{\sqrt{|C_2|}} \sum_{w \in C_2} |w+v\rangle : v \in C_1 \right\}$$

$$Q^* = \left\{ \frac{1}{\sqrt{|C_1^{\perp}|}} \sum_{w \in C_1^{\perp}} |w+v\rangle : v \in C_2^{\perp} \right\}$$

CSS Q-ECCs

Let $C_1=[n,k_1,d_1]$, $C_2^{\perp}=[n,n-k_2,d_2]$ be two linear codes

$$\dim(Q) = \dim(C_1)\text{-}\dim(C_2^{\perp}) = k_1-k_2 = \dim(C_2)\text{-}\dim(C_1^{\perp}) = \dim(Q^*)$$

$$d(Q) = d(Q^*) = \min\{d(C_1),d(C_2^{\perp})\} = \min\{d_1,d_2\}$$

$$Q = [[\ n,\ k_1-k_2,\ \min\{d_1,d_2\}\ ]] = Q^*$$

CSS Q-ECCs

EXAMPLE: Quantum Reed-Solomon codes (Aharonov-BenOr)

Let $q=4t$

$C_1 = [4t,2t+1,2t]$ ERS-code over GF(q)
$C_2 = [4t,2t,2t+1]$ ERS-code over GF(q)

$\dim(Q) = \dim(Q^*) = 1$
$d(Q) = d(Q^*) = 2t$

$Q,Q^*= [[4t,\ 1,\ 2t]]$ QRS-code over GF(q)

$Q,Q^*= [[n,\ 1,\ n/2]]$ QRS-code over GF(q), $q=n$

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