Isoperimetric Inequalities and Eigenvalues

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Abstract

An upper bound is given on the minimum distance between i subsets of the same size of a regular graph in terms of the i-th largest eigenvalue in absolute value. This yields a bound on the diameter in terms of the i-th largest eigenvalue, for any integer i. Our bounds are shown to be asymptotically tight. A recent result by Quenell relating the diameter, the second eigenvalue, and the girth of a regular graph is obtained as a byproduct.

Key words. eigenvalues, diameter, Chebychev polynomials, expanders

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1 Introduction

Many combinatorial properties of a graph are related to the spectrum of its adjacency matrix [2, 3, 4, 18]. The adjacency matrix A of an undirected graph is the $0-1$ matrix indexed by the vertices, and such that the entry (u, v) is equal to 1 if and only if (u, v) is an edge. Since the adjacency matrix of any graph H on n vertices is symmetric and real, its eigenvalues are real and will be denoted by $\lambda_0(H) \geq \lambda_1(H) \geq \cdots \geq \lambda_{n-1}(H)$. In this paper, we explore the relation between the spectrum of a graph and its isoperimetric properties. We focus our attention on the diameter, which is defined to be the maximum distance in H between any pair of vertices, and will be denoted by $D(H)$. The diameter plays an important role in network design, in parallel and distributed computing.

Let $\lambda = \lambda(H) = \max(\lambda_1, |\lambda_{n-1}|)$. It is known that if a graph is k-regular, then $\lambda_0 = k$ and $\lambda \leq k$, with equality if and only if the graph is disconnected or bipartite. Moreover, the graph is an expander if and only if [2] there exists a gap between k and λ_1. Thus, the existence of an upper

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bound on the diameter in terms of the eigenvalue gap is not surprising. Such a bound first appeared in [3], where it was shown that, when G is k-regular,

$$D(G) \leq 2\sqrt{\frac{2k}{(k-\lambda_1)}}\log_2 n. \tag{1}$$

Chung [6] established that

$$D(G) \leq \left\lfloor \frac{\log(n-1)}{\log(k/\lambda)} \right\rfloor + 1, \tag{2}$$

which beats Eq. 1 when λ is small. Eq. 2 was further improved in [7, 16, 17], where it was shown that

$$D(G) \leq \left\lfloor \frac{\cosh^{-1}(n-1)}{\cosh^{-1}(k/\lambda)} \right\rfloor + 1. \tag{3}$$

For fixed n and k, the right-hand side is small when λ is small. It is known, however, that for any sequence $G_{n,k}$ of k-regular graphs on n vertices, $\liminf \lambda(G_{n,k}) \geq 2\sqrt{k-1}$ as n goes to infinity [2, 13, 15]. A Ramanujan graph is a k-regular graph where all eigenvalues not equal to $\pm k$ are at most $2\sqrt{k-1}$ in absolute value. Such graphs have been constructed explicitly in [13, 14]. Note that when $\lambda = 2\sqrt{k-1}$, the right-hand side of Eq. 3 is roughly $2\log_{k-1} n$.

In this paper, we establish some isoperimetric bounds that are function of the subsequent eigenvalues and do not depend on the second eigenvalue. More precisely,

Theorem 1 *Let $G = (V, E)$ be an undirected k-regular graph, and $\delta_0, \delta_1, \ldots, \delta_{n-1}$ the eigenvalues of its adjacency matrix, with $|\delta_0| \geq |\delta_1| \geq \ldots \geq |\delta_{n-1}|$. If $X_1, X_2, \ldots, X_{i+1}$ are $i+1$ subsets of V of same cardinality xn, then*

$$\min_{1 \leq j < h \leq i+1} d(X_j, X_h) \leq \left\lfloor \frac{\cosh^{-1}(x^{-1}-1)}{\cosh^{-1}(k/|\delta_i|)} \right\rfloor + 2.$$

■

Eq. 2 is (up to an additive constant 1) a particular case of Theorem 1. We will use Theorem 1 to derive upper bounds on the diameter of G in terms of δ_i. In Section 3, we establish a lower bound on the size of $N^t(X)$, where X is a set of vertices and $N^t(X)$ is the set of nodes that can be reached from X by a path of length t, that is,

$$\underbrace{N(N(\ldots N(}_{t \text{ times}} X))\ldots).$$

The lower bound on $|N^t(X)|$ is a function of the size of X and of the second eigenvalue in absolute value λ of the graph. As a first corollary, we obtain an upper bound on the distance between two subsets of given size. As a second corollary, we get a simple proof of a recent result [16] relating the diameter, the girth, and λ. In Section 4, we prove Theorem 1 and derive a relation between the

diameter of a graph and its subsequent eigenvalues. In Section 5, we show that Eq. 2 is tight by exhibiting a family of k-regular graphs with $\lambda = (2 + o(1))\sqrt{k-1}$ and diameter $(2 + o(1)) \log_{k-1} n$. Since our graphs have asymptotically optimal second eigenvalue, this shows that the bound $2 \log_{k-1} n$ is the best one can obtain using the second eigenvalue method. Note that the diameter of a random k-regular graph on n vertices [5, p. 238] is almost surely $(1 + o(1)) \log_{k-1} n$. We also prove that our bounds on the diameter in terms of δ_i are asymptotically tight, for any fixed i.

Our proofs are based on elementary linear algebra, the use of Chebychev polynomials, and some techniques developed in [10, 11, 12]. Section 3 is based on [9], and a longer version of the paper appears in [11].

2 Notation and background

Let $G = (V, E)$ be an undirected k-regular graph on n vertices. Denote by $L^2(V)$ the set of real valued functions on V and $L_0^2(V) = \{f \in L^2(V);\ \sum_{v \in V} f(v) = 0\}$. As usual, we define the scalar product of two vectors f and g of $L^2(V)$ by

$$f \cdot g = \sum_{v \in V} f(v) g(v),$$

and the euclidean norm of a vector f by $||f|| = \sqrt{f \cdot f}$. The adjacency matrix A of G defines a linear operator in $L^2(V)$ that maps every vector $f \in L^2(V)$ to the vector Af defined by

$$(Af)(v) = \sum_{(v,w) \in E} f(w). \tag{4}$$

This operator is selfadjoint since $\forall f, g \in L^2(V)$, we have

$$(Af) \cdot g = f \cdot (Ag) = \sum_{(v,w) \in E} f(v) g(w). \tag{5}$$

For any subset W of V, we denote by χ_W the characteristic vector of W:

$$\chi_W(v) = \begin{cases} 1 & \text{if } v \in W \\ 0 & \text{otherwise} \end{cases}$$

The support of a vector $f \in L^2(V)$ is defined to be the set of nodes v for which $f(v) \neq 0$. We sometimes order the eigenvalues of a graph H according to their absolute values and denote them by $\delta_i(H)$, so that $|\delta_0(H)| \geq |\delta_1(H)| \geq \cdots \geq |\delta_{n-1}(H)|$. For any matrix B with real eigenvalues, we denote its $(i+1)$-st largest eigenvalue by $\lambda_i(B)$.

Fact 1 *If L is a vector space and B is a selfadjoint operator in L whose largest (resp. smallest) eigenvalue in absolute value is γ (resp. δ), then $|\delta|\ ||g|| \leq ||Bg|| \leq |\gamma|\ ||g||$ for any $g \in L$.* ■

Fact 2 *If B is a selfadjoint operator in a vector space L, then*

$$\lambda_i(B) = \max_H \min_{g \in H - \{0\}} \frac{g \cdot Bg}{||g||^2},$$

where H ranges over the vector subspaces of L of dimension $i+1$. ■

Fact 3 *For any operator B and any polynomial P, the eigenvalues of the operator $P(B)$ are the images by P of the eigenvalues of B.* ■

Fact 4 *The eigenvalues of a graph are the union of the eigenvalues of its connected components.* ■

Let P_t be the Chebychev polynomial of degree t. It is the unique polynomial satisfying the equation

$$P_t(\cos y) = \cos(ty), \tag{6}$$

for any complex number y. The Chebychev polynomials have already been used in [13] in the study of expanders. The following facts follow easily from Eq. 6:

Fact 5 *For any complex number z, we have $P_t(-z) = (-1)^t P_t(z)$.* ■

Fact 6 *For any real number s between -1 and 1, we have $|P_t(s)| \leq 1$* ■

Fact 7 *For any complex number z, we have $P_t(\cosh z) = \cosh(tz)$.* ■

The l^1-norm $||h||_1$ of a vector h is defined to be $\sum_{v \in V} |h(v)|$.

Fact 8 *For any $h \in L^2(V)$, we have*

$$||h|| \leq ||h||_1 \leq \sqrt{n}||h||.$$

■

3 Bounds on the distance between two subsets

We start by proving lower bounds on $|N^t(X)|$.

Theorem 2 *Let $G = (V, E)$ be a k-regular graph on n vertices and λ its second largest eigenvalue in absolute value. For any subset X of V and any integer $t \geq 1$, we have*

$$|N^t(X)| \geq \frac{{P_t}^2(k/\lambda)|X|}{1 + ({P_t}^2(k/\lambda) - 1)|X|/n}. \tag{7}$$

If G is a non-bipartite Ramanujan graph of degree k, then

$$\frac{|N^t(X)|}{|X|} \geq \frac{((k-1)^t+1)^2}{4(k-1)^t + ((k-1)^t-1)^2|X|/n} \geq \frac{(k-1)^t}{4+(k-1)^t|X|/n}.$$

In particular, if $|X|/n$ is at most $4(k-1)^{-t-1}$, then $|N^t(X)| \geq (k-2)(k-1)^{t-1}|X|/4$.

Proof Denote by f be the characteristic vector of X. Let $f = \bar{f} + f_0$, where $\bar{f}$ is a constant vector and $f_0 \in L_0^2(V)$. The parity property of the polynomial P_t shows that it is of the form $P_t(s) = c_t s^t + c_{t-2}s^{t-2} + \cdots$, and so $P_t(\lambda^{-1}A)\ f = c_t\lambda^{-t}A^t f + c_{t-2}\lambda^{-(t-2)}A^{t-2}f + \ldots$. This implies that the support of the vector $g = P_t(\lambda^{-1}A)\ f$ is a subset of $N^t(X)$. This is because the support of the vector $A^l f$ is $N^l(X)$, and $N^t(X) \supseteq N^{t-2}(X) \supseteq \ldots$. We will obtain a lower bound on the size of $N^t(X)$ by comparing the norm of g and its sum of coordinates. Since $A\bar{f} = k\bar{f}$, we have

$$g = P_t(\lambda^{-1}A)\ \bar{f} + P_t(\lambda^{-1}A)\ f_0 = P_t(k/\lambda)\ \bar{f} + P_t(\lambda^{-1}A)\ f_0.$$

Eq. 4 shows that $L_0^2(V)$ is invariant under A. Hence, $P_t(\lambda^{-1}A)\ f_0 \in L_0^2(V)$. Since A is selfadjoint, it is diagonalizable in an orthonormal basis. The eigenvector corresponding to k is constant, and so the eigenvectors corresponding to λ_i, for $1 \leq i \leq n-1$, belong to $L_0^2(V)$. Therefore, the eigenvalues of the restriction $A_{|L_0^2(V)}$ of A to $L_0^2(V)$ are λ_i, for $1 \leq i \leq n-1$. By the Pythagorean theorem, we have

$$||g||^2 = {P_t}^2(k/\lambda)||\bar{f}||^2 + ||P_t(\lambda^{-1}A)\ f_0||^2 \leq {P_t}^2(k/\lambda)||\bar{f}||^2 + ||f_0||^2.$$

The second inequality follows from the fact that the operator $P_t(\lambda^{-1}A_{|L_0^2(V)})$ is selfadjoint and its eigenvalues $P_t(\lambda_i/\lambda)$, $1 \leq i \leq n-1$, are at most 1 in absolute value (Facts 3, 6 and 1.) It follows from the Cauchy-Schwartz inequality that

$$\begin{aligned} |\chi_{N^t(X)} \cdot g|^2 &\leq |N^t(X)|\ ({P_t}^2(k/\lambda)||\bar{f}||^2 + ||f_0||^2) \\ &= |N^t(X)|\ ({P_t}^2(k/\lambda)||\bar{f}||^2 + |X| - ||\bar{f}||^2). \end{aligned} \tag{8}$$

The second equation follows by applying the Pythagorean theorem to the vector $f = \bar{f} + f_0$. Since the operator $P_t(\lambda^{-1}A)$ is selfadjoint and the support of g is a subset of $N^t(X)$, we have

$$\begin{aligned} \chi_{N^t(X)} \cdot g &= \chi_V \cdot (P_t(\lambda^{-1}A)\ f) \\ &= (P_t(\lambda^{-1}A)\ \chi_V) \cdot f \\ &= P_t(k/\lambda)\ \chi_V \cdot f \\ &= P_t(k/\lambda)|X|. \end{aligned}$$

By replacing the terms $\chi_{N^t(X)} \cdot g$ and $||\bar{f}||$ by their values in Eq. 8, we get

$$\frac{|X|}{|N^t(X)|} \leq \frac{|X|}{n} + \frac{1 - |X|/n}{{P_t}^2(k/\lambda)}, \tag{9}$$

which implies Eq. 7.

If G is a non-bipartite Ramanujan graph, we can replace λ by $2\sqrt{k-1}$ in Eq 7. We have

$$\begin{aligned} P_t(\frac{k}{2\sqrt{k-1}}) &= P_t(\cosh(\frac{\ln(k-1)}{2})) \\ &= \cosh(t\frac{\ln(k-1)}{2}) \\ &= \frac{(k-1)^{t/2} + (k-1)^{-t/2}}{2}. \end{aligned}$$

The rest of the theorem follows by an easy calculation. ■

Corollary 1 *If X and Y are two subsets of V of cardinality xn and yn respectively,*

$$d(X,Y) \leq \left\lfloor \frac{\cosh^{-1}\sqrt{(x^{-1}-1)(y^{-1}-1)}}{\cosh^{-1}(k/\lambda)} \right\rfloor + 1.$$

Proof If t is an integer such that the right-hand side of Eq. 9 is less than $|X|/(n-|Y|)$, then $|N^t(X)| > n - |Y|$, and so the distance between X and Y is at most t. Let $\theta = \cosh^{-1}(k/\lambda)$, so that $P_t(k/\lambda) = \cosh(t\theta)$. We want t to be such that

$$x + \frac{1-x}{\cosh^2(t\theta)} < \frac{x}{1-y},$$

which is equivalent to

$$\cosh^2(t\theta) > \frac{(1-x)(1-y)}{xy}$$

Solving for t yields the desired bound on $d(X,Y)$. ■

By applying Corollary 1 to any pair of subsets consisting of single vertices, we obtain Eq. 3, which has already been established in [7, 17].

Corollary 2 ([16])

$$D(G) \leq \left\lfloor \frac{\cosh^{-1}\left(\frac{n}{k(k-1)^{r-1}} - 1\right)}{\cosh^{-1}(k/\lambda)} \right\rfloor + 2r + 1,$$

where $r = \lfloor (c(G)-1)/2 \rfloor$ is the injectivity radius *of G.*

Proof We remind the reader that $c(G)$ is the girth of G. Consider any pair of vertices u and v. The subsets $N^r(\{u\})$ and $N^r(\{v\})$ have size $k(k-1)^{r-1}$. By applying Corollary 1 to these subsets,

we get

$$d(N^r(\{u\}, N^r(\{v\}) \leq \left\lfloor \frac{\cosh^{-1}\left(\frac{n}{k(k-1)^{r-1}} - 1\right)}{\cosh^{-1}(k/\lambda)} \right\rfloor + 1.$$

Corollary 2 follows immediately. ■

Corollary 2 has first been established by Quenell [16], who argued that it always gives a better bound than Eq. 3.

4 Relation with subsequent eigenvalues

We generalize Corollary 1 as follows:

Theorem 3 *If $G = (V, E)$ is a k-regular graph, then for any set S of $i+1$ vertices of V,*

$$\min_{\{u,v\} \subset S, u \neq v} \leq \left\lfloor \frac{\cosh^{-1}(n-1)}{\cosh^{-1}(k/|\delta_i|)} \right\rfloor + 2.$$

Proof Let e_j be an eigenvector of A corresponding to the eigenvalue δ_j, and let $f \in L^2(V)$ be a nonzero function nul on $V - S$, and such that f belongs to the vector space E_i spanned by $e_i, e_{i+1}, \ldots, e_{n-1}$. The existence of f follows from the fact that $\dim L^2(S) = i+1$ and $\dim E_i = n-i$. Given an integer t, let $g = P_t(|\delta_i|^{-1}A)f$. The vector space E_i is invariant under A, and the eigenvalues of the restriction of A to E_i are δ_h, for $i \leq h \leq n-1$. By a similar reasoning to the proof of Theorem 2, we know that the eigenvalues $P_t(\delta_h/|\delta_i|)$, for $i \leq h \leq n-1$, of the restriction of the operator $P_t(|\delta_i|^{-1}A)$ to E, are at most 1 in absolute value, and so $||g|| \leq ||f||$. Assume now that $\min_{\{u,v\} \subset S, u \neq v} d(u,v) > 2t$. Then the vectors $P_t(|\delta_i|^{-1}A)\chi_{\{u\}}$, for $u \in S$, have disjoint supports, and so

$$\begin{aligned}
||g||_1 &= ||\sum_{u \in S} f(u)\ P_t(|\delta_i|^{-1}A)\chi_{\{u\}}||_1 \qquad (10)\\
&= \sum_{u \in S} |f(u)|\ ||P_t(|\delta_i|^{-1}A)\chi_{\{u\}}||_1 \\
&\geq \sum_{u \in S} |f(u)|\ (P_t(|\delta_i|^{-1}A)\chi_{\{u\}}) \cdot \chi_V \\
&= \sum_{u \in S} |f(u)|\ (P_t(|\delta_i|^{-1}A)\chi_V) \cdot \chi_{\{u\}} \\
&= \sum_{u \in S} |f(u)| P_t(k/|\delta_i|) \\
&= P_t(k/|\delta_i|)||f||_1.
\end{aligned}$$

On the other hand,

$$\begin{aligned} ||g||_1 &\leq \sqrt{n}||g|| \\ &\leq \sqrt{n}||f|| \\ &\leq \sqrt{n/2}||f||_1. \end{aligned} \tag{11}$$

The first inequality follows from Fact 8. The last inequality is valid because $f \in L_0^2(V)$. Indeed,

$$\begin{aligned} ||f||^2 &= ||f^+||^2 + ||f^-||^2 \\ &\leq ||f^+||_1^2 + ||f^-||_1^2 \\ &= \frac{||f||_1^2}{2}, \end{aligned}$$

where $f^+ = \max(f, 0)$ and $f^- = \min(f, 0)$.

Comparing Eq. 10 and Eq. 11 shows that $P_t(k/|\delta_i|) \leq \sqrt{n/2}$. If

$$t = \left\lfloor \frac{\cosh^{-1}\sqrt{n/2}}{\cosh^{-1}(k/|\delta_i|)} \right\rfloor + 1,$$

the above inequality does not hold, and so $\min_{\{u,v\}\subset S, u\neq v} \leq 2t$. The lemma follows by noting that $2\cosh^{-1}\sqrt{n/2} = \cosh^{-1}(n-1)$. ■

Theorem 1 can be shown in a similar way to Theorem 3. The main difference is that we consider a function f in $L^2(\cup_{j=1}^{i+1} X_j)$ which is constant on each X_j.

Theorem 4 *If G is a k-regular connected graph, then for any integer i between 1 and $n-1$,*

$$D(G) \leq i \left\lfloor \frac{\cosh^{-1}(n-1)}{\cosh^{-1}(k/|\delta_i|)} \right\rfloor + 3i - 1. \tag{12}$$

If r is the injectivity radius of G, then for any integer i between 1 and $n-1$,

$$D(G) \leq i \left\lfloor \frac{\cosh^{-1}(\frac{n}{k(k-1)^{r-1}} - 1)}{\cosh^{-1}(k/|\delta_i|)} \right\rfloor + 2ir + 3i - 1.$$

Proof Let u and v be two vertices at maximal distance in G. Consider a shortest path between u and v. There exists a sequence of $i+1$ vertices $u_0 = u, u_1, \ldots, u_i = v$ on this path at distance at least $\lfloor D(G)/i \rfloor$ from each other. By applying Theorem 3 to the set $\{u_0, u_1, \ldots, u_i\}$, we get the first bound on $D(G) = d(u, v)$.

The second bound can be established similarly by applying Theorem 1 to the subsets $N^r(\{u_j\})$.

■

5 Tightness of bounds

We use techniques similar to [10, 11, 12] to prove that the inequality Eq. 12 is asymptotically tight, for any fixed i. We start with the case $i = 1$.

Theorem 5 *For any integer k such that $k-1$ is prime congruent to 1 modulo 4, there exists an infinite family of k-regular graphs G_n on n vertices with $\lambda(G_n) = (2+o(1))\sqrt{k-1}$ and diameter $(2+o(1))\log_{k-1} n$.*

Proof We start with a k-regular Ramanujan graph H on n' vertices, and we form two identical trees T and T' of depth $l = \lfloor \log_{k-1} m - 2 \rfloor$, where $m = \lfloor n'/\log n' \rfloor$, and whose internal nodes have degree k. All leaves in T and T' have the same depth. Consider a set E^r of edges in H at distance $r = \Omega(\log_{k-1}(n'/m))$ from each other and having the same cardinality as the number of leaves in T. For each edge in E^r, identify one of its endpoints to a leaf in T and the other endpoint to a leaf in T', so that the leaves of T and T' are identified to distinct vertices in H. By deleting the edges in E^r, we obtain a k-regular graph $G = G_n$. The diameter of G is at least twice the depth of T, which is $(1+o(1))\log_{k-1} n$. We will show that $\lambda(G) = (2+o(1))\sqrt{k-1}$. Eq. 3 then implies that the diameter of G is equal to $(2+o(1))\log_{k-1} n$. Let A be the adjacency matrix of H, A' the adjacency matrix of G, and $\lambda' = \lambda(G)$. We assume that $\lambda' > 2\sqrt{k-1}$ (otherwise, we are done), and let $\lambda' = 2\sqrt{k-1}\cosh\theta'$, with $\theta' > 0$. We also assume that $\lambda' = \lambda_1(G)$. The case $\lambda' = -\lambda_{n-1}(G)$ can be treated similarly. Denote by $V(G)$, $V(H)$, $V(T)$ and $V(T')$ the vertex sets of G, H, T and T', respectively.

Let $g \in L^2(V(G))$ be an eigenvector of A' corresponding to λ', and let $f \in L^2(V(H))$ be the vector of $L^2(H)$ that coincides with g on H. By Eq. 5, we have

$$\begin{aligned} \lambda'||g||^2 &= g \cdot A'g \qquad (13)\\ &= f \cdot Af - \sum_{(u,v)\in E^r} g(u)g(v) + \sum_{(u,v)\in E(T)} g(u)g(v) + \sum_{(u,v)\in E(T')} g(u)g(v) \\ &\le f \cdot Af + (2\sqrt{k-1}+1) \sum_{v\in V(T)\cup V(T')} g(v)^2. \end{aligned}$$

The third equation follows from the fact that the largest eigenvalue of T is at most $2\sqrt{k-1}$ and the largest eigenvalue of the edge-subgraph E^r is 1. Since $g \in L_0^2(V(G))$,

$$\sum_{w\in V(H)} f(w) = - \sum_{w\in (V(T)\cup V(T'))-V(H)} g(w).$$

We need the following lemma, whose proof can be found in [10, 11, 12].

Lemma 1 *If $G = (V, E)$ is k-regular on n vertices, then for any $f \in L^2(V)$, we have*

$$f \cdot Af \leq \lambda_1(G)||f||^2 + \frac{k - \lambda_1(G)}{n}\Big(\sum_{v \in V} f(v)\Big)^2.$$

■

Using Lemma 1 and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} f \cdot Af &\leq \lambda_1(G)||f||^2 + \frac{k}{n}\Big(\sum_{w \in (V(T) \cup V(T')) - V(H)} g(w)\Big)^2 \\ &\leq 2\sqrt{k-1}(||g||^2 - \sum_{w \in (V(T) \cup V(T')) - V(H)} g(w)^2) + \frac{2km}{n} \sum_{w \in (V(T) \cup V(T')) - V(H)} g(w)^2 \\ &\leq 2\sqrt{k-1}||g||^2, \end{aligned}$$

for sufficiently large n. Combining this with Eq. 13 yields

$$\lambda'||g||^2 \leq 2\sqrt{k-1}(||g||^2 + 2 \sum_{v \in V(T) \cup V(T')} g(v)^2). \tag{14}$$

Next, we show that $\sum_{v \in V(T) \cup V(T')} g(v)^2$ is small compared to $||g||^2$. We use the following lemma, whose proof is implicit in [10, 12], and given in detail in [11].

Lemma 2 *Let $G = (V, E)$ be a k-regular graph and g an eigenvector of G corresponding to the eigenvalue $2\sqrt{k-1}\cosh\theta$, with $\theta > 0$. If l and l' are two nonegative integers with $l < l'$, and u is a node of G such that the subgraph induced on the set of nodes at distance at most l' from u is a tree, then*

$$\frac{\sum_{v \in V : d(u,v) = l'} g(v)^2}{\sum_{v \in V : d(u,v) = l} g(v)^2} \geq e^{2(l' - l)\theta}.$$

■

By applying Lemma 2 in the case where u is the root of T and $l' = l + r$, we see that

$$||g||^2 \geq e^{2r\theta'} \sum_{v \in V : d(u,v) = l} g(v)^2.$$

By applying the lemma to $l - 1, l - 2, \ldots, 0$, we obtain

$$||g||^2 \geq e^{2r\theta'}(1 - e^{-2\theta'}) \sum_{v \in T} g(v)^2.$$

Combining this Eq. 14 yields

$$\cosh\theta' \leq 1 + 2\frac{e^{-2r\theta'}}{1 - e^{-2\theta'}}.$$

As a consequence, $\theta' \leq 3(\log r)/(2r)$, for large n, and so $\lambda' \leq (2+o(1))\sqrt{k-1}$. On the other hand [2, 13, 15], $\lambda' \geq (2+o(1))\sqrt{k-1}$ for any family of k-regular graphs, as the number of vertices goes to infinity. ■

The following theorem was obtained jointly with Noga Alon [1].

Theorem 6 *If $k-1$ is a prime congruent to 1 modulo 4 and i is a positive integer, there exists an infinite family of k-regular graphs G_n on n vertices of diameter $(2+o(1))i\log_{k-1} n$ and such that $\delta_j(G_n) = k - O(1/n)$, for $0 \leq j \leq i-1$, and $|\delta_i(G_n)| = (2+o(1))\sqrt{k-1}$.*

Proof Consider a family (F_n) of k-regular graphs satisfying the conditions of Theorem 5. We construct the graphs G_n (for n multiple of i and such that $F_{n/i}$ exists) as follows: consider i distinct copies of $F_{n/i}$, denoted by $F^j_{n/i}$, for $1 \leq j \leq i$. Let (u_j, v_j) be a pair of vertices in $F^j_{n/i}$ at maximal distance from each other, and let u'_j (resp. v'_j) be a vertex of $F^j_{n/i}$ adjacent to u_j (resp. v_j). We form the graph G_n by connecting v_j (resp. v'_j) to u_{j+1} (resp. u'_{j+1}), for $1 \leq j \leq i-1$, and deleting the edge between u_j and u'_j, for $2 \leq j \leq i$, and the edge between v_j and v'_j, for $1 \leq j \leq i-1$. Clearly, the diameter of G_n is $(2+o(1))i\log_{k-1} n$.

The eigenvalues of the union of the $F_{n/i}$ satisfy the conditions of Theorem 6. Indeed, by Fact 4, the first i eigenvalues are equal to k, and the $(i+1)$-st largest eigenvalue in absolute value is, in absolute value, equal to $\lambda(F_{n/i}) = (2+o(1))\sqrt{k-1}$. We will show that the $i+1$ largest eigenvalues of G_n are close to the $i+1$ largest eigenvalues of the union of the $F_{n/i}$.

Let A (resp. A') be the adjacency matrix of the union of the $F_{n/i}$ (resp. G_n.) Let V_j be the vertex set of $F^j_{n/i}$, and V the vertex set of G_n. It follows from Eq. 5 that for any $f \in L^2(V)$,

$$\begin{aligned} |A'f\cdot f - Af\cdot f| &= 2\Big|\sum_{j=1}^{i-1}(f(v_j)f(u_{j+1}) + f(v'_j)f(u'_{j+1}) - f(u_{j+1})f(u'_{j+1}) - f(v_j)f(v'_j))\Big| \quad (15)\\ &\leq 2\sum_{j=1}^{i}(f(u_j)^2 + f(u'_j)^2 + f(v_j)^2 + f(v'_j)^2). \end{aligned}$$

Denote by H be the subspace of $L^2(V)$ of vectors constant on each V_j. Eq. 15 shows that, for each $f \in H$, we have $A'f\cdot f \geq (k-8i/n)||f||^2$. Since $\dim H = i$, it follows from Fact 2 that $\lambda_{i-1}(G)$ is lower bounded by $k-8i/n$, and so are $|\delta_0|, |\delta_1|, \ldots, |\delta_{i-1}|$.

We know [8] that $|\delta_i(G_n)| \geq (2+o(1))\sqrt{k-1}$ for any family of k-regular graphs, as the number of vertices goes to infinity. So we only need to show that $|\delta_i| \leq (2+o(1))\sqrt{k-1}$. Let r be the injectivity radius of G_n. It is at least equal to the injectivity radius of $F_{n/i}$, which is $\Omega(\log_{k-1}\log n)$. If $|\delta_i| \leq 2\sqrt{k-1}$, we are done. So we will assume in the rest of the proof that $\delta_i > 2\sqrt{k-1}$ (the case where $\delta_i < -2\sqrt{k-1}$ can be treated similarly.) Let $\delta_i = 2\sqrt{k-1}\cosh\theta$, with $\theta > 0$. Let e_h be an eigenvector of A' corresponding to δ_h, for $0 \leq h \leq i$. Since $|\delta_h| \geq \delta_i$, for $0 \leq j \leq i$, it follows from Lemma 2 that $e^{r\theta}|e_h(u_j)| \leq ||e_h||$. Using the Cauchy-Schwartz inequality and the orthogonality of

the vectors e_h, this implies that for any vector $f \in \mathrm{Vect}(e_0, e_1, \ldots, e_i)$,

$$e^{r\theta}|f(u_j)| \leq \sqrt{i+1}||f||. \tag{16}$$

Indeed, if $f = \sum_{h=0}^{i} c_h e_h$, then

$$\begin{aligned} f(u_j)^2 &\leq (i+1)\sum_{h=0}^{i} {c_h}^2 e_h(u_j)^2 \\ &\leq \frac{i+1}{e^{2r\theta}}\sum_{h=0}^{i} {c_h}^2||e_h||^2 \\ &= \frac{i+1}{e^{2r\theta}}||f||^2. \end{aligned}$$

Eq. 16 remains valid if u_j is replaced by u'_j, v_j or v'_j. Since the vector space $\mathrm{Vect}(e_0, e_1, \ldots, e_i)$ has greater dimension than H, it intersects $H^{\perp} - \{0\}$. Let g be an element of this intersection. Since the restriction of g to each V_j belongs to $L_0^2(V_j)$, it follows from Fact 1 that

$$||Ag||^2 \leq \lambda(F_{n/i})^2||g||^2 \leq (4 + o(1))(k-1)||g||^2.$$

Combining this with Eq. 16 applied to the vector $f = A'g$ yields

$$\begin{aligned} ||A'g||^2 &\leq ||Ag||^2 + \sum_{j=1}^{i}((A'g)(u_j)^2 + (A'g)(u'_j)^2 + (A'g)(v_j)^2 + (A'g)(v'_j)^2) \\ &\leq (4 + o(1))(k-1)||g||^2 + \frac{4i(i+1)}{e^{2r\theta}}||A'g||^2. \end{aligned} \tag{17}$$

But, since $g \in \mathrm{Vect}(e_0, e_1, \ldots, e_i)$, we have $||A'g||^2 \geq {\delta_i}^2||g||^2$ (Fact 1.) Combining this with Eq. 17 shows that

$$\cosh^2\theta \leq 1 + o(1) + \frac{4i(i+1)\cosh^2\theta}{e^{2r\theta}},$$

which implies that $\theta = o(1)$. Replacing δ_i by its value, we conclude that $|\delta_i| \leq (2 + o(1))\sqrt{k-1}$. ■

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