

Big-Oh, Big-Ω, Big-Θ

Let g be a function from the natural integers to the positive reals,
 $g : \aleph \rightarrow \mathbb{R}^+$.

$$O(g) = \{ f : \aleph \rightarrow \mathbb{R}^+ \mid \exists c > 0, \exists n_0 > 0, n \geq n_0 \Rightarrow f(n) \leq c g(n) \}$$

$$\Omega(g) = \{ f : \aleph \rightarrow \mathbb{R}^+ \mid \exists c > 0, \exists n_0 > 0, n \geq n_0 \Rightarrow f(n) \geq c g(n) \}$$

$$f \text{ is } \Omega(g) \Leftrightarrow g \text{ is } O(f)$$

$$\begin{aligned}\Theta(g) &= \{ f : \aleph \rightarrow \mathbb{R}^+ \mid f \text{ is } \Omega(g) \text{ and } f \text{ is } O(g) \} \\ &= O(g) \cap \Omega(g)\end{aligned}$$

Big-Oh, Big-Ω, Big-Θ

$n \in O(n^2)$ (n is $O(n^2)$) since for $c=1$, $n_0=1$, $n \geq n_0 \Rightarrow n \leq n^2$

$n^2 \notin O(n)$ (n^2 is not $O(n)$) since for any c , $n_0 > c$, $n \geq n_0 \Rightarrow n^2 > cn$

$$O(n) \subset O(n^2)$$

$n^{100} \in O(10^n)$ since for $c=1$, $n_0=300$, $n \geq n_0 \Rightarrow n^{100} \leq 10^n$

$f_n \in O(\phi^n)$ since for $c=2/\sqrt{5}$, $n_0=1$, $n \geq n_0 \Rightarrow f_n \leq 2\phi^n/\sqrt{5}$

$n^2 \in \Omega(n)$ since for $c=1$, $n_0=2$, $n \geq n_0 \Rightarrow n^2 \geq n$

$an+b \in \Theta(n)$ since $an+b \in O(n)$ and $an+b \in \Omega(n)$

Big-Oh rules

Let $d, e, f, g : \mathbb{N} \rightarrow \mathbb{R}^+$.

1. If $d \in O(f)$, then $a^*d \in O(f)$, for any constant $a > 0$
2. If $d \in O(f)$ and $e \in O(g)$, then $d+e \in O(f+g)$
3. If $d \in O(f)$ and $e \in O(g)$, then $d^*e \in O(f^*g)$
4. If $d \in O(f)$ and $f \in O(g)$, then $d \in O(g)$
5. If f is a polynomial of degree d , then $f \in O(n^d)$
6. $n^x \in O(a^n)$ for any fixed $x > 0$, $a > 1$
7. $\log n^x \in O(\log n)$ for any fixed $x > 0$
8. $(\log n)^x \in O(n^y)$ for any fixed $x > 0$, $y > 0$

Big-Oh limit rules

Let $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$.

1. If $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$, then $f \in O(g)$ and $g \notin O(f)$,

$$O(f) \subset O(g)$$

2. If $\lim_{n \rightarrow \infty} f(n)/g(n) = c > 0$, then $f \in O(g)$ and $g \in O(f)$,

$$O(f) = O(g)$$

3. If $\lim_{n \rightarrow \infty} f(n)/g(n) = \infty$, then $f \notin O(g)$ and $g \in O(f)$,

$$O(f) \supset O(g)$$