Welcome to...

Convex Hell

er, that's

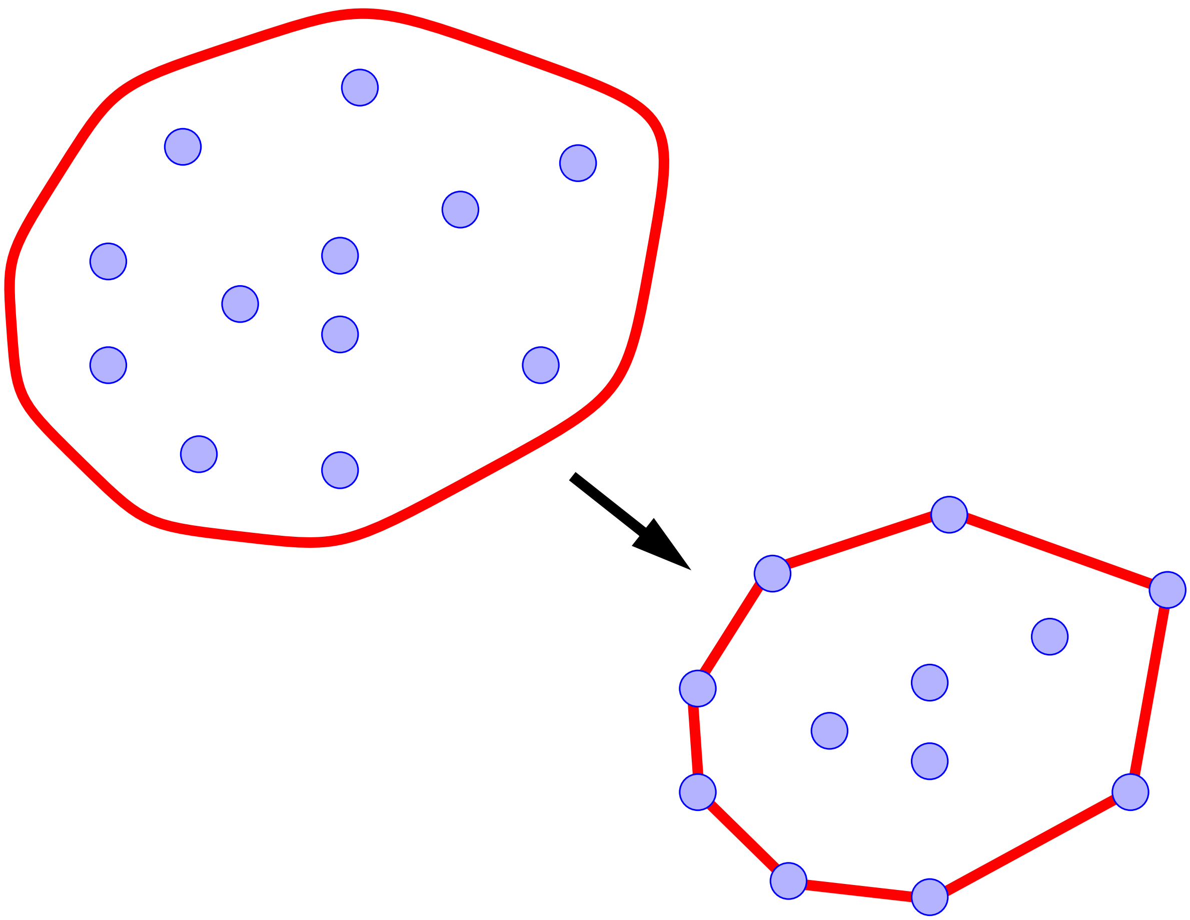
Convex HULL

- Convexivity
- Package-Wrap Algorithm
- Graham Scan
- Dynamic Convex Hull

What is the Convex Hull?

Let ***S*** be a set of points in the plane.

Intuition: Imagine the points of ***S*** as being pegs; the ***convex hull*** of ***S*** is the shape of a rubber-band stretched around the pegs.



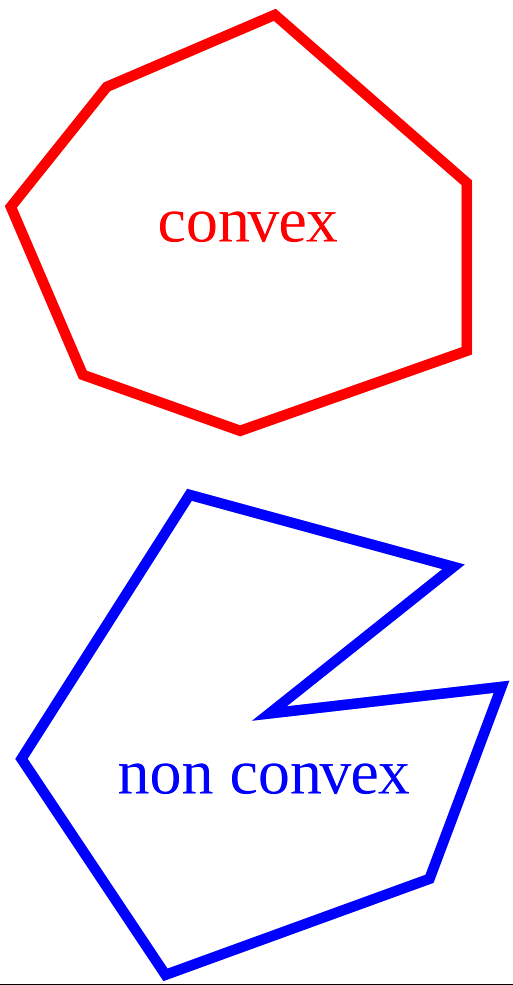
Formal definition: the ***convex hull*** of ***S*** is the smallest convex polygon that contains all the points of ***S***

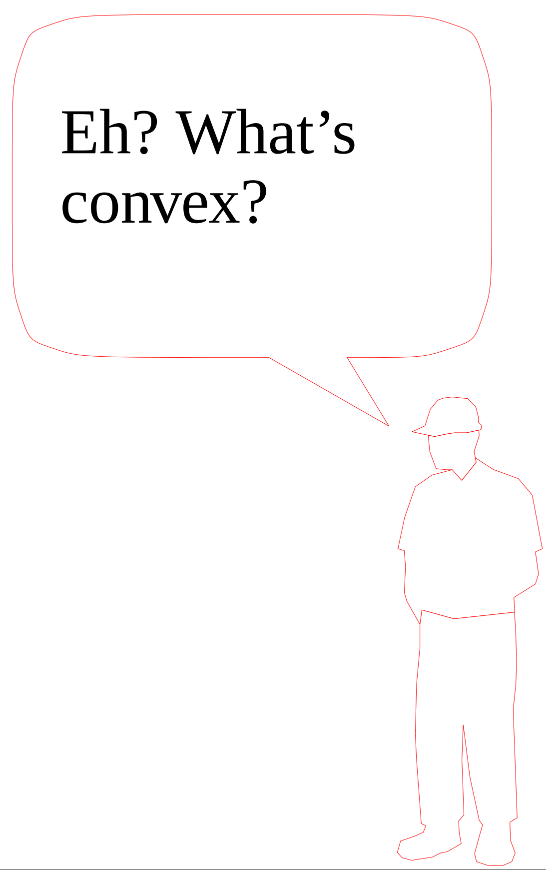
Convexity

You know what ***convex*** means, right?

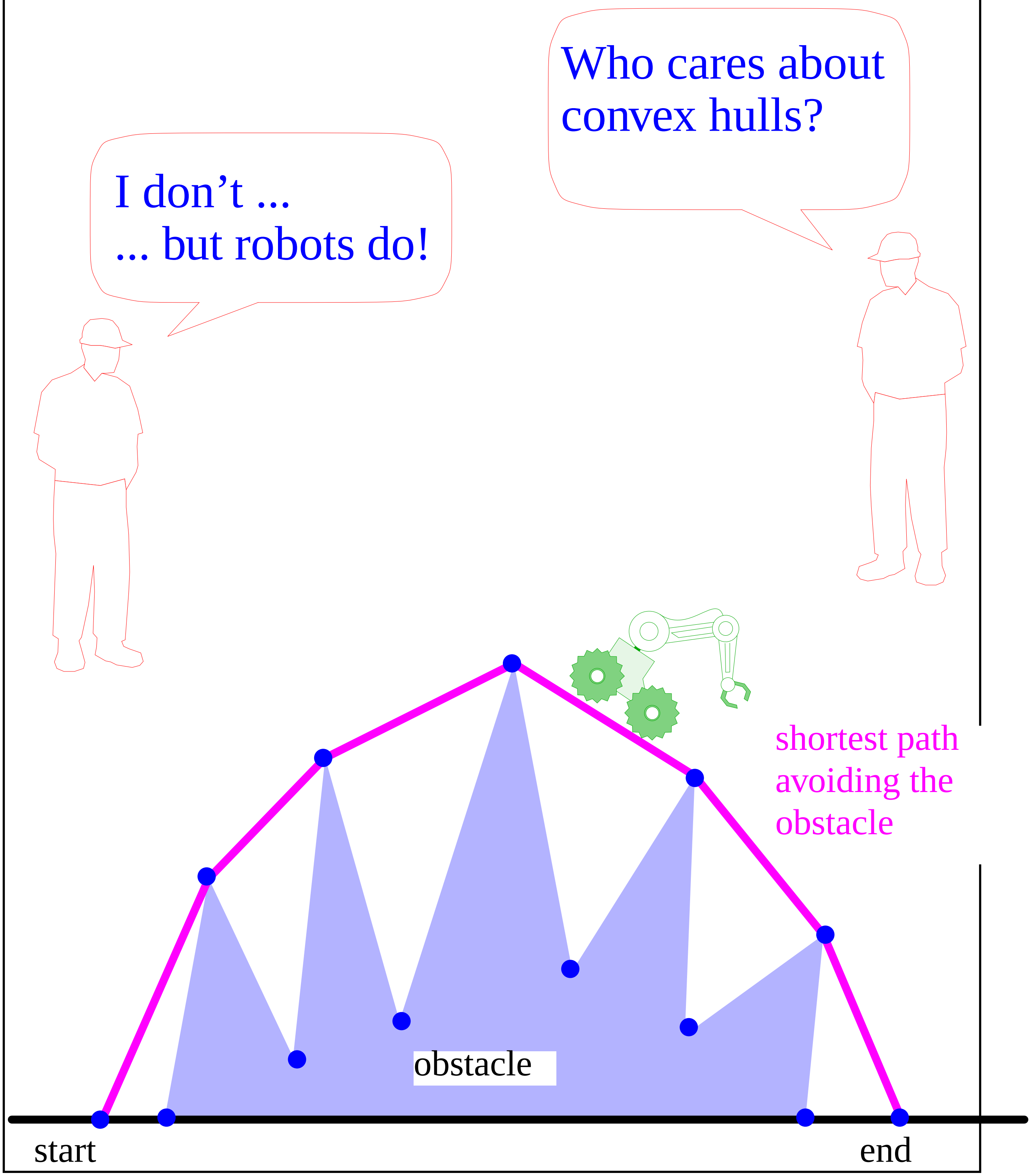
A polygon ***P*** is said to be ***convex*** if:

1. ***P*** is non-intersecting; and
2. for any two points ***p*** and ***q*** on the boundary of ***P***, segment ***pq*** lies entirely inside ***P***

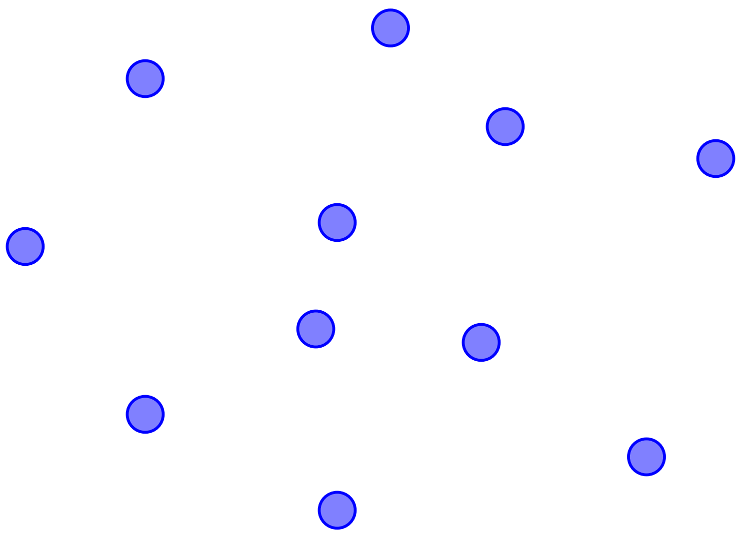




Why Convex Hulls?



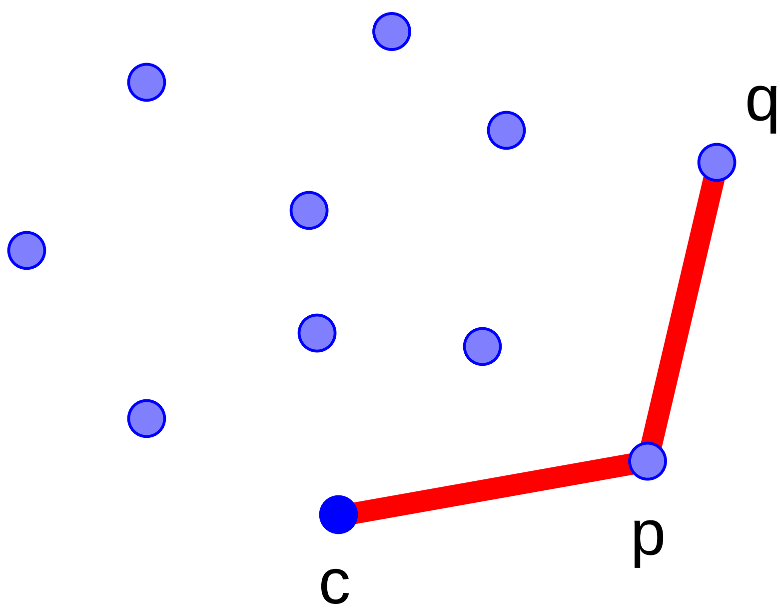
The Package Wrapping Algorithm



Package Wrap

- given the current point, how do we compute the next point?
- set up an orientation tournament using the current point as the anchor-point...
- the next point is selected as the point that beats all other points at CCW orientation, i.e., for any other point, we have

orientation(c, p, q) = CCW



Time Complexity of Package Wrap

- For every point on the hull we examine all the other points to determine the next point
- Notation:
  - N: number of points
  - M: number of hull points ($M \leq N$)
- Time complexity:
  - $\Theta(MN)$
- Worst case: $\Theta(N^2)$
  - all the points are on the hull ($M=N$)
- Average case: $\Theta(N \log N)$ — $\Theta(N^{4/3})$
  - for points randomly distributed inside a ***square***, $M = \Theta(\log N)$ on average
  - for points randomly distributed inside a ***circle***, $M = \Theta(N^{1/3})$ on average

Package Wrap has worst-case time complexity O(N^2)

Which is bad...

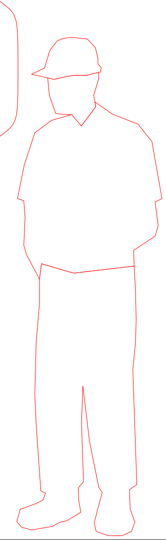
But in 1972, Nabisco needed a better cookie - so they hired R. L. Graham, who came up with...

The Graham Scan Algorithm

Rave Reviews:

- “Almost linear!”
  - Sedgewick
- “It’s just a sort!”
  - Atul
- “Two thumbs up!”
  - Siskel and Ebert
- Nabisco says...

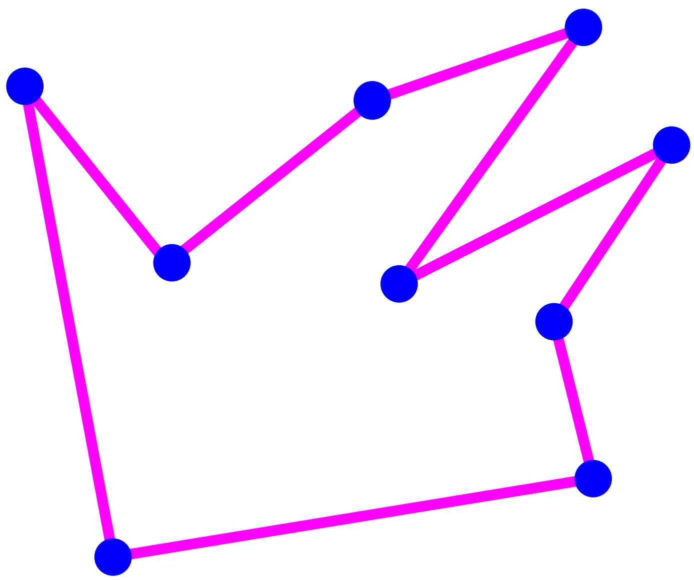
“A better crunch!”



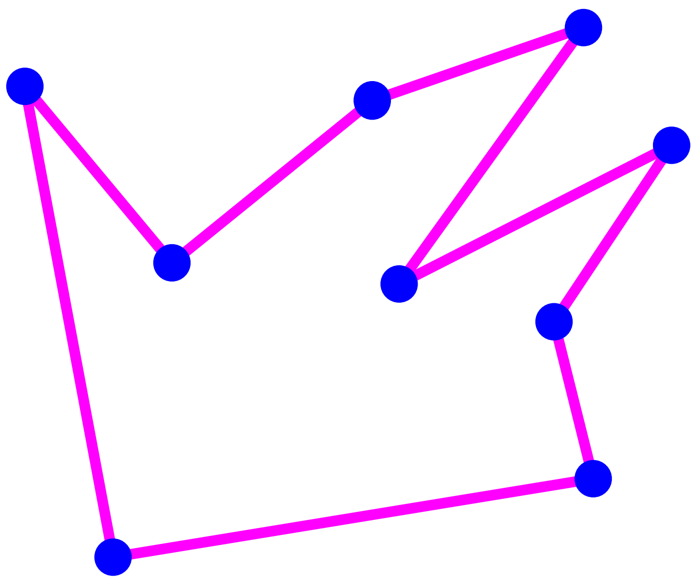
and history was made.

Graham Scan

- Form a simple polygon (connect the dots as before)

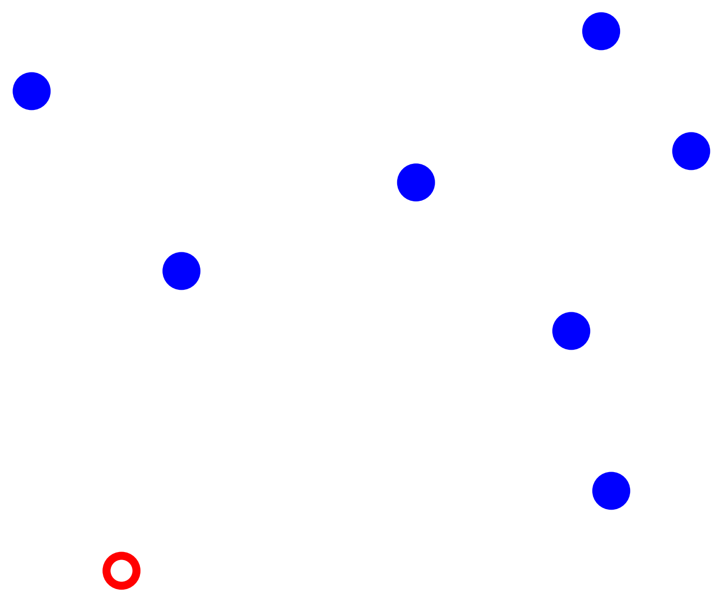


- Remove points at concave angles



Graham Scan

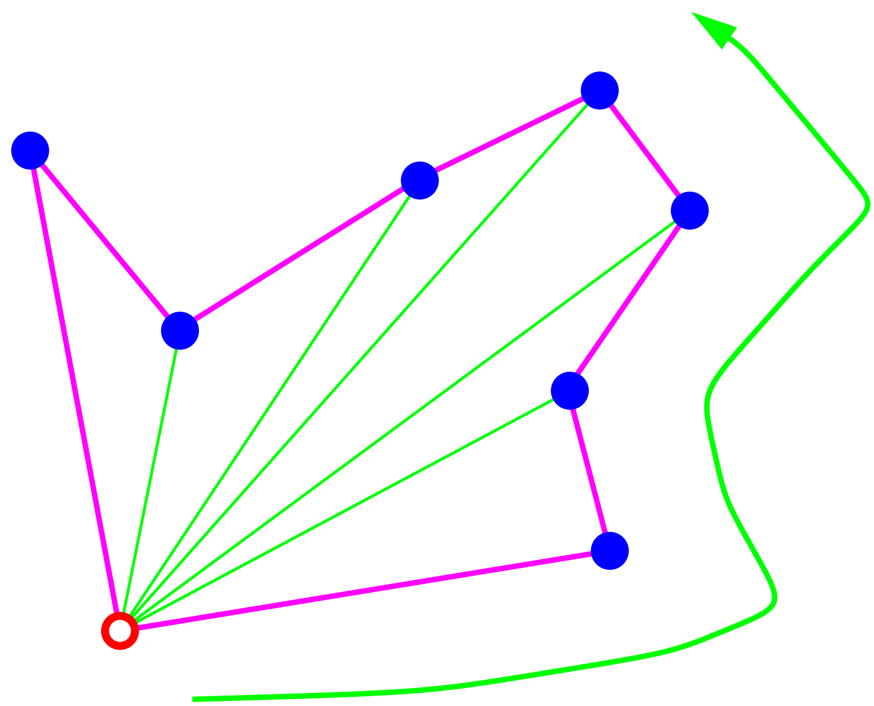
How Does it Work?



Start with the lowest point (anchor point)

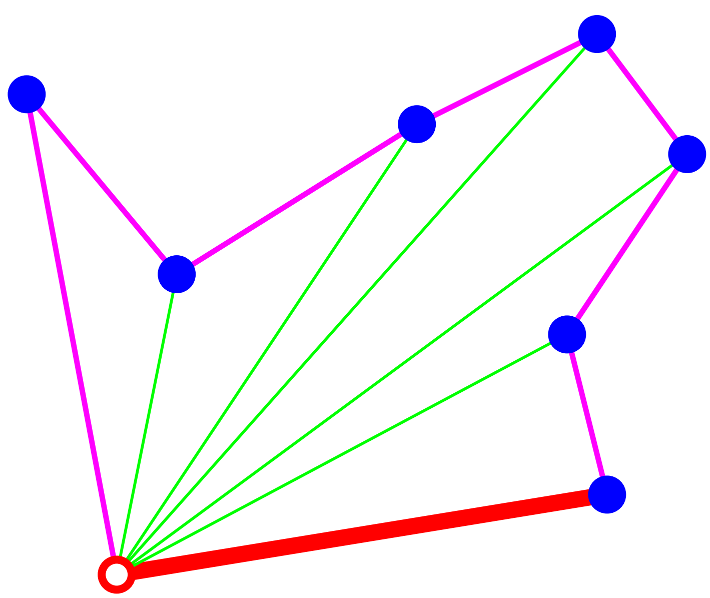
Graham Scan: Phase 1

Now, form a closed simple path traversing the points by increasing angle with respect to the anchor point



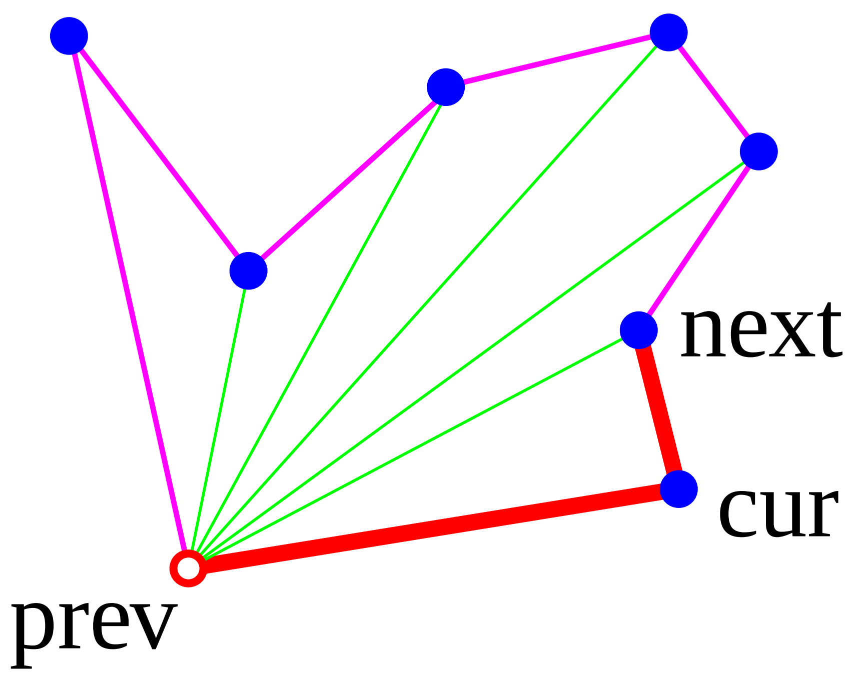
Graham Scan: Phase 2

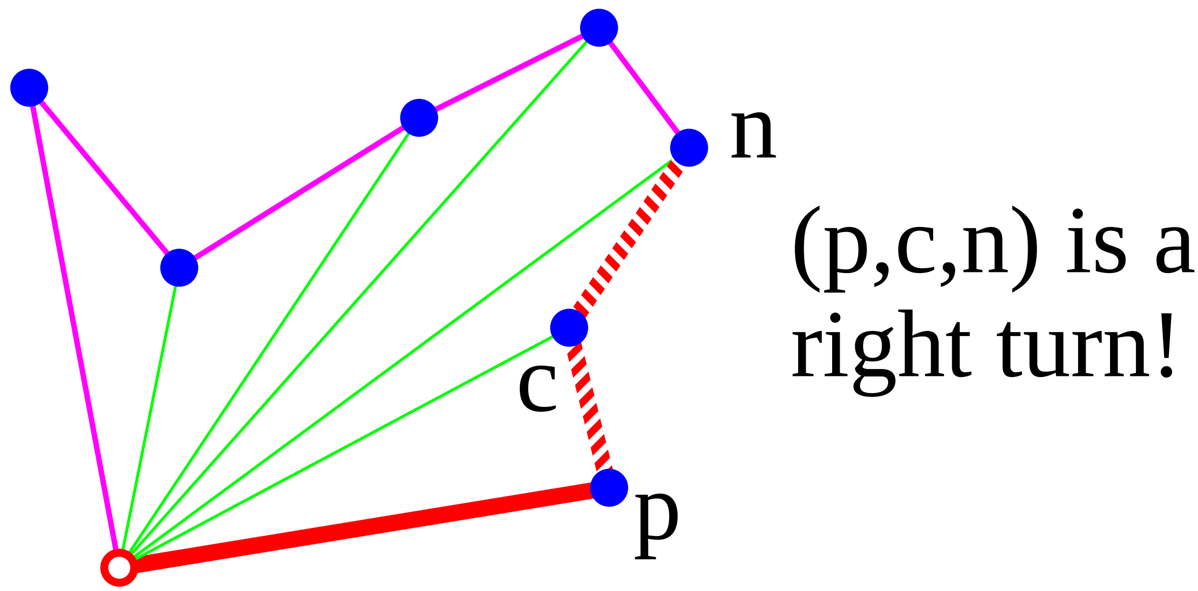
The anchor point and the next point on the path must be on the hull (why?)



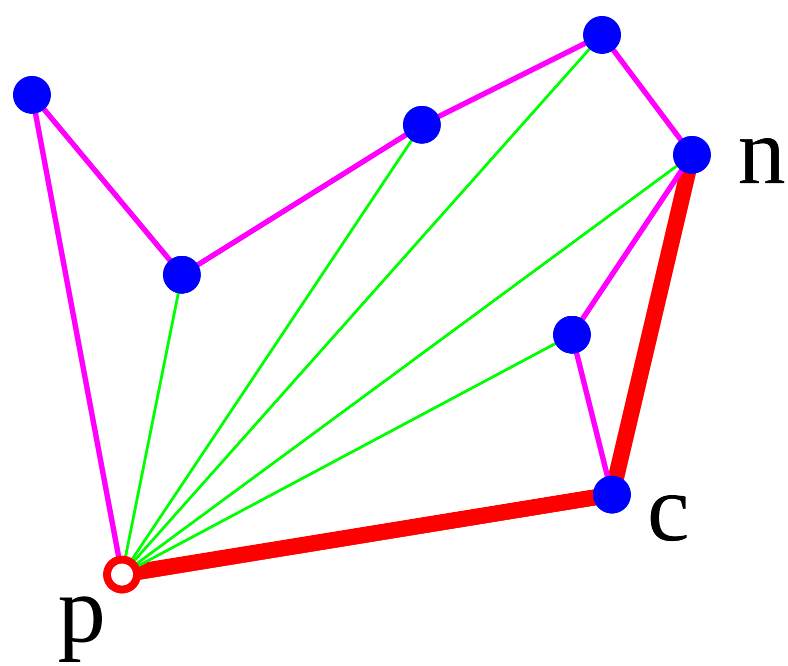
Graham Scan: Phase 2

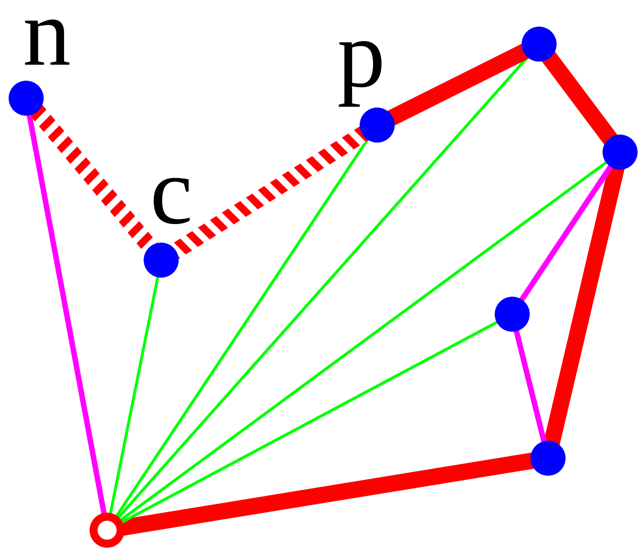
- keep the path and the hull points in two sequences
- elements are removed from the beginning of the path sequence and are inserted and deleted from the end of the hull sequence
- orientation is used to decide whether to accept or reject the next point



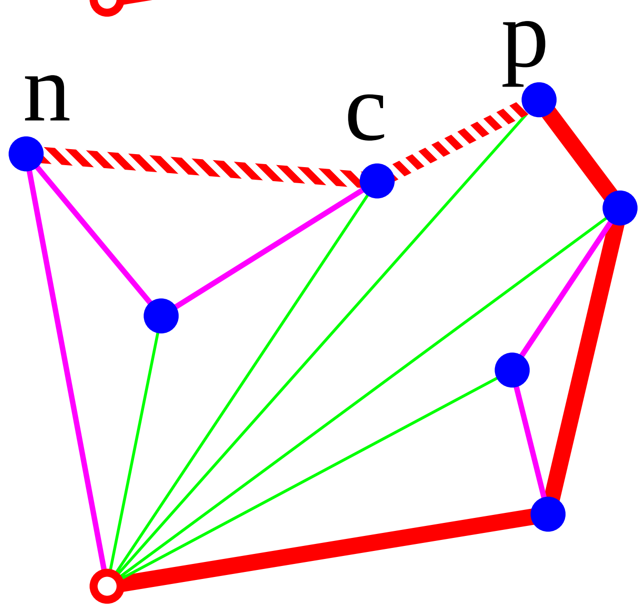


Discard c





(p,c,n) is a right turn!



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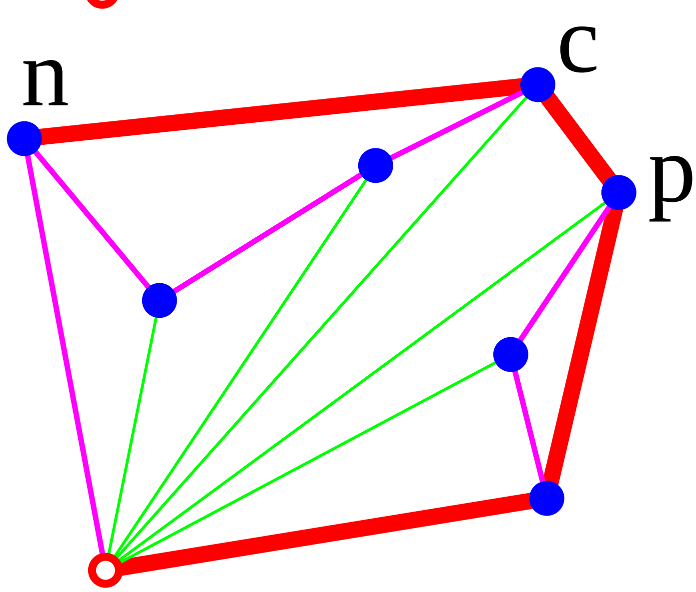
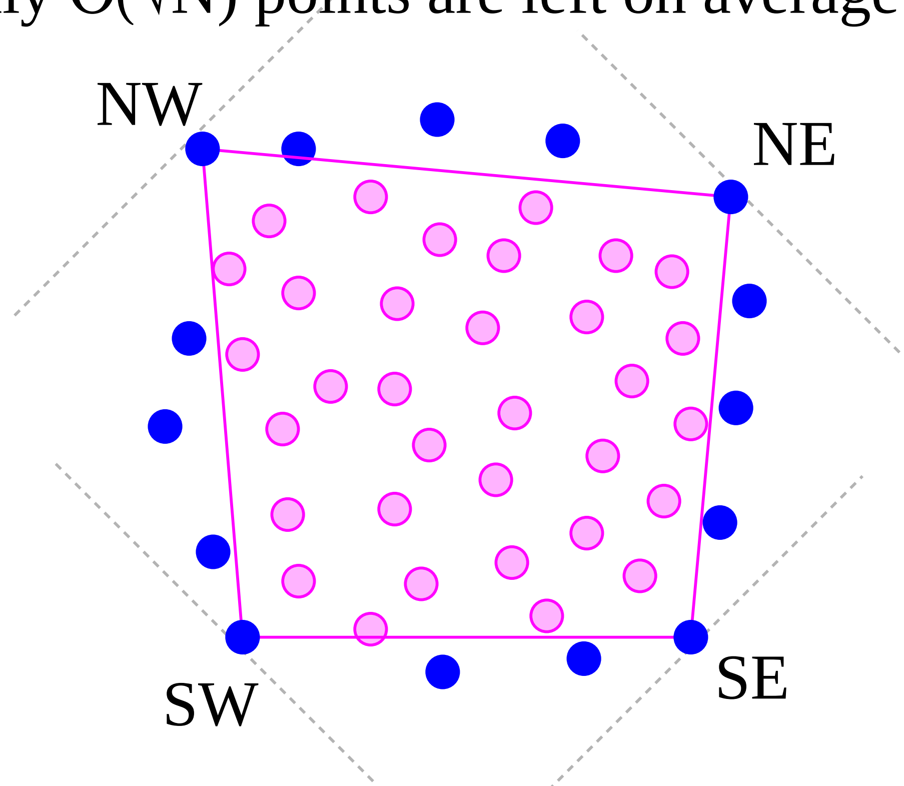


Time Complexity of Graham Scan

- Phase 1 takes time O(N logN)
  - points are sorted by angle around the anchor
- Phase 2 takes time O(N)
  - each point is inserted into the sequence exactly once, and
  - each point is removed from the sequence at most once
- Total time complexity O(N log N)

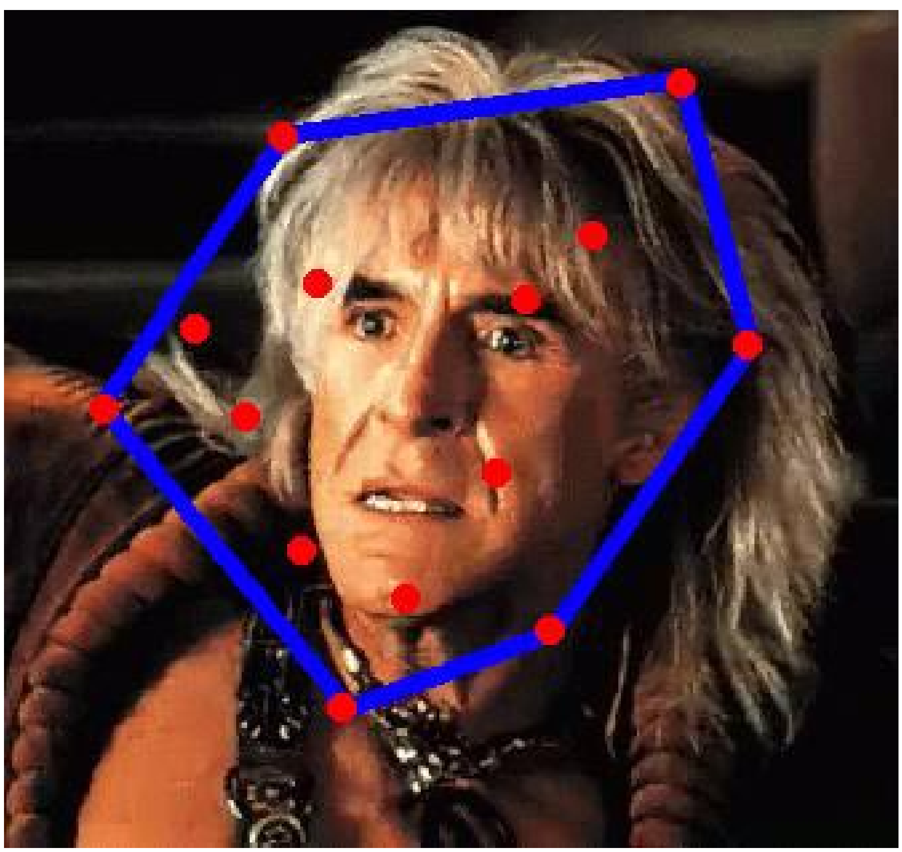
How to Increase Speed

- Wipe out a lot of the points you know won't be on the hull! This is ***interior elimination***
- Here's a good way to do interior elimination if the points are randomly distributed in a square with horizontal and verticall sides:
  - Find the farthest points in the SW, NW, NE, and SE directions
  - Eliminate the points inside the quadrilateral (SW, NW, NE, SE)
  - Do Package Wrap or Graham Scan on the remaining points (only O(√N) points are left on average!)



Dynamic Convex Hull a.k.a. the Wrath of Khan

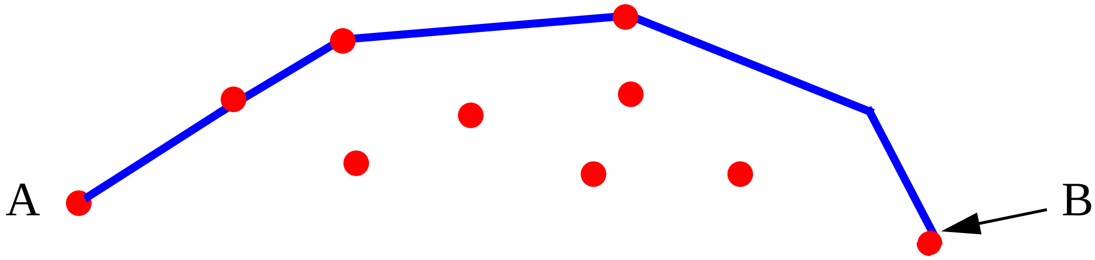
- The basic convex hull algorithms were fairly interesting, but you may have noticed that you can't draw the hull until after all of the points have been specified.
- Is there an *interactive* way to add points to the hull and redraw it while maintaining an optimal time complexity?



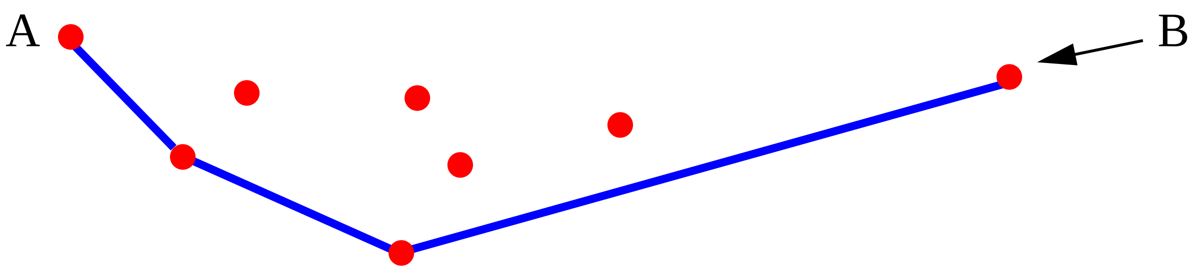
Two Halves = One Hull

- For this algorithm, we consider a convex hull as being two parts:

  - An upper hull:

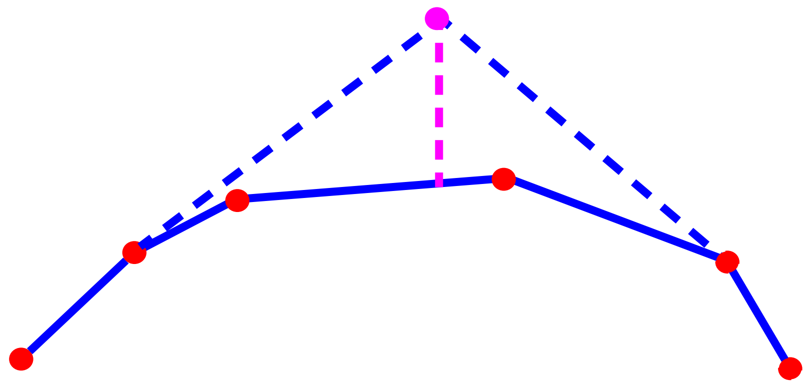


  - and a lower hull:

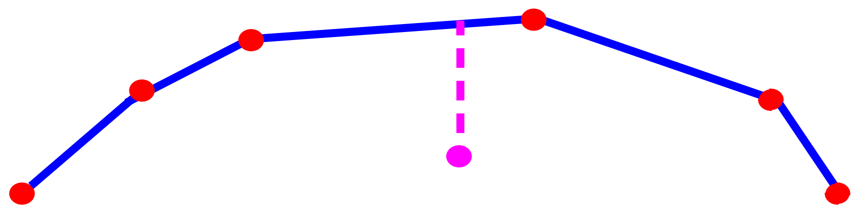


Adding points: Case 1

- Case 1: the new point is within the horizontal span of the hull
  - Upper Hull 1a:
    If the new point is above the upper hull, then it should be added to the upper hull and some upper-hull points may be removed.



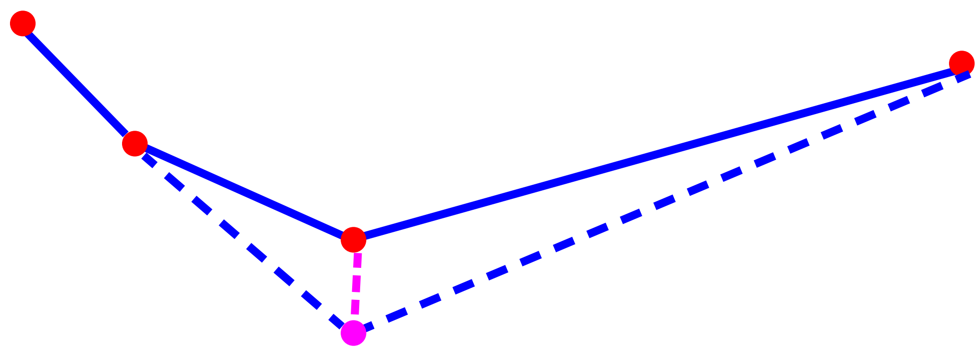
  - Upper Hull 1b:
    If the new point is below the upper hull, no changes need to be made.



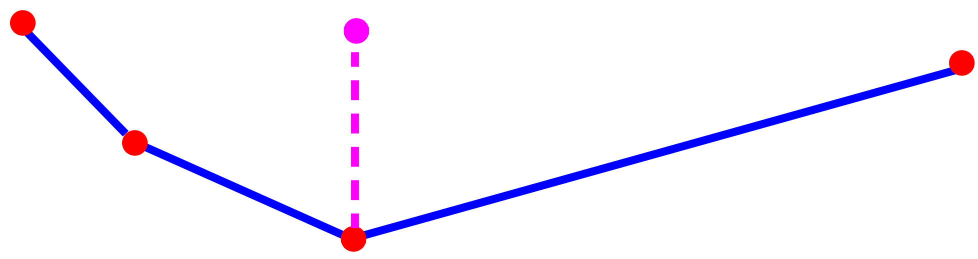
Case 1 (cont.)

- The cases for the lower hull are similar.

  - Lower Hull 1a:
    If the new point is below the lower hull, then it is added to the lower hull and some lower-hull points may be removed.

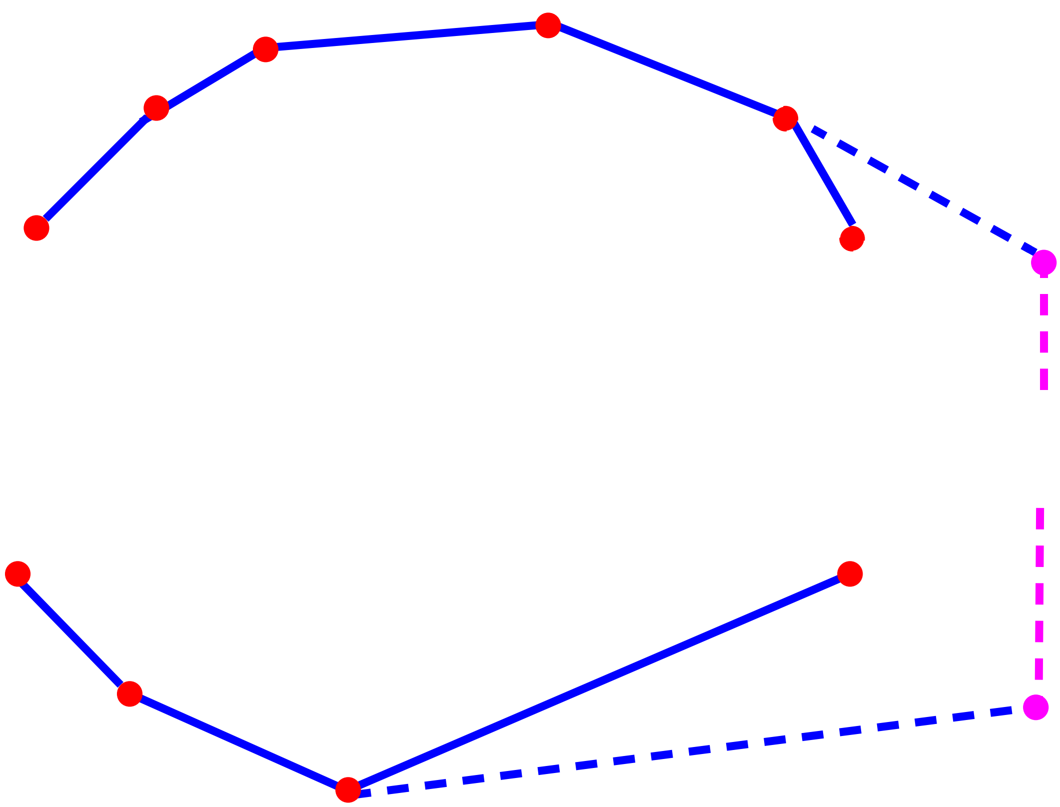


  - Lower Hull 1b:
    If the added point is above the existing point, it is inside the existing lower hull, and no changes need be made.



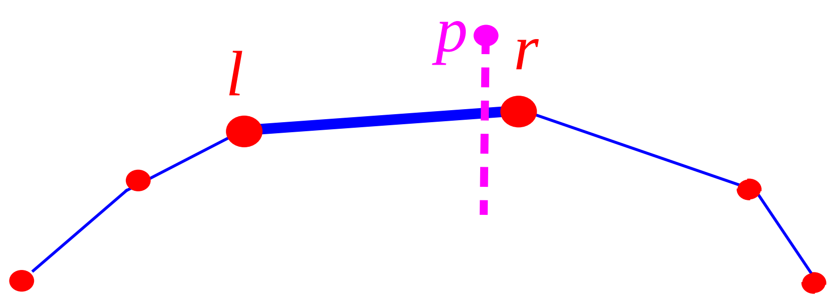
Adding Points: Case 2

- Case 2: the new point is outside the horizontal span of the hull
  - We must modify both the upper and lower hulls accordingly.



Hull Modification

- In Case 1, we determine the vertices *l* and *r* of the upper or lower hulls immediately preceding/following the new point *p* in the *x*-order.



- If *p* has been added to the upper hull, examine the upper hull rightward starting at *r*. If *p* makes a CCW turn with *r* and its right neighbor, remove *r*. Continue until there are no more CCW turns. Repeat for point *l* examining the upper hull leftward. The computation for the bottom hull is similar.