Trees

Graphs :

A *graph* G is a pair (V, E) :

- V is a finite set called the *vertex set* of G. Its elements are called *vertices*.

- E is a binary relation on V called the *edge set* of G. Its elements are called *edges*.

A graph can either be *directed* or *undirected*. In a *directed* graph, *edges* are represented by arrows. For example, the pair (u, v) represents an edge—arrow going from vertex u to vertex v which would be different then an edge going from v to u, (v, u). Note that *self-loops* – edges from a vertex to itself – are possible in directed graphs.

These self-loops are forbidden in *undirected* graphs, graphs where edges are *unordered* pairs of vertices rather than ordered pairs. In these (u, v) is the same edge as (v, u).

If (u, v) is an edge in a directed graph $G = (V, E)$, we say the edge *leaves* the vertex u to *enter* the vertex v. Moreover, this vertex v is said to be *adjacent* to vertex u.

- The *degree* of a vertex in an undirected graph is the number of edges incident on it.

- A *path* of length k, from a vertex u to a vertex u' in a graph $G = (V, E)$, is a sequence $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ of vertices such that $u = v_0, u' = v_k$ and all egdes (v_{i-1}, v_i) used are elements of E, the set of edges. The length of the path is the number of edges in the path.

- A path is *simple* if all the vertices in the path are distincts.

- A *cycle* is a path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ where $v_0 = v_k$ that contains at least one edge.

- An undirected *cycle* is a simple path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ where $v_0 = v_k$ that contains at least three edges.

- A graph with no *cycle* is called *acyclic*.

- An undirected graph is *connected* if every pair of vertices is connected by a path.

An <u>acyclic</u>, <u>undirected</u> <u>graph</u> is a *forest*. A <u>connected</u> *forest* is called a *free tree*. Since we will return to graphs later on, let us now concentrate on these *trees*

Properties of trees :

Let $G = (V, E)$ be an undirected graph. The following statements are equivalent.

1. G is a free tree.
   $\Downarrow$
2. Any two vertices in G are connected by a unique simple path.
   $\Downarrow$
3. G is connected, but if any edge is removed from E, the resulting graph is disconnected.
   $\Downarrow$
4. G is connected, and $|E| = |V| - 1$.
   $\Downarrow$
5. G is acyclic, and $|E| = |V| - 1$.
   $\Downarrow$
6. G is acyclic, but if any edge is added to E, the resulting graph contains a cycle.

Show 6 $\Rightarrow$ 1

Rooted tree :

A *rooted tree* is a tree in which one of the vertices is distinguished from the others. This vertex is called the root. Vertices of a tree are usually called nodes.

Consider a node x in a rooted tree T with root r.

- A node y on the unique path from r to x is called an *ancestor* of x.

- x is then called a descendant of y.

- The *subtree rooted at* x is the tree induced by the descendants of x.

- If y is the direct ancestor of x such that (x, y) is an edge of G, then y is the *parent* of x and x is the child of y.

- The root is the only node in T with no parents.

- A node with no children is an *external node* or *leaf*. A nonleaf node is an *internal leaf*.

- The number of children of a node x in a rooted T is called the *degree* of x. Note that in a free tree or in an undirected graph, the *degree* of a node is the number of adjacent vertices. In a rooted tree, the *degree* is the number of children – the parent does not count.

- A *k-ary* tree is a tree in which each node has a maximum of k children, each identified uniquely from 1 to k. We say the ith child of a node is absent if no child is labeled with integer i.

- A k-ary tree of value $k = 2$ is called a *binary tree*.

- A *complete* k-ary tree is a tree in which all leaves have the same depth and all internal nodes have degree k.

  The number of internal nodes of a complete k-ary tree of height h is

  $$\begin{aligned} 1 + k + k^2 + \ldots + k^{h-1} &= \sum_{i=0}^{h-1} k^i \\ &= \frac{k^h - 1}{k - 1} \end{aligned}$$

  Thus, a complete binary tree has $2^h - 1$ internal nodes.

- The length of the path from the root r to a node x is the depth of x in T.

- The largest depth of any node in T is the *height* of T.