from Quantum Coding to Quantum Authenticating, and back

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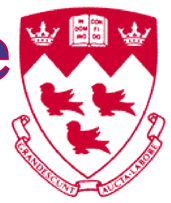


joint work with

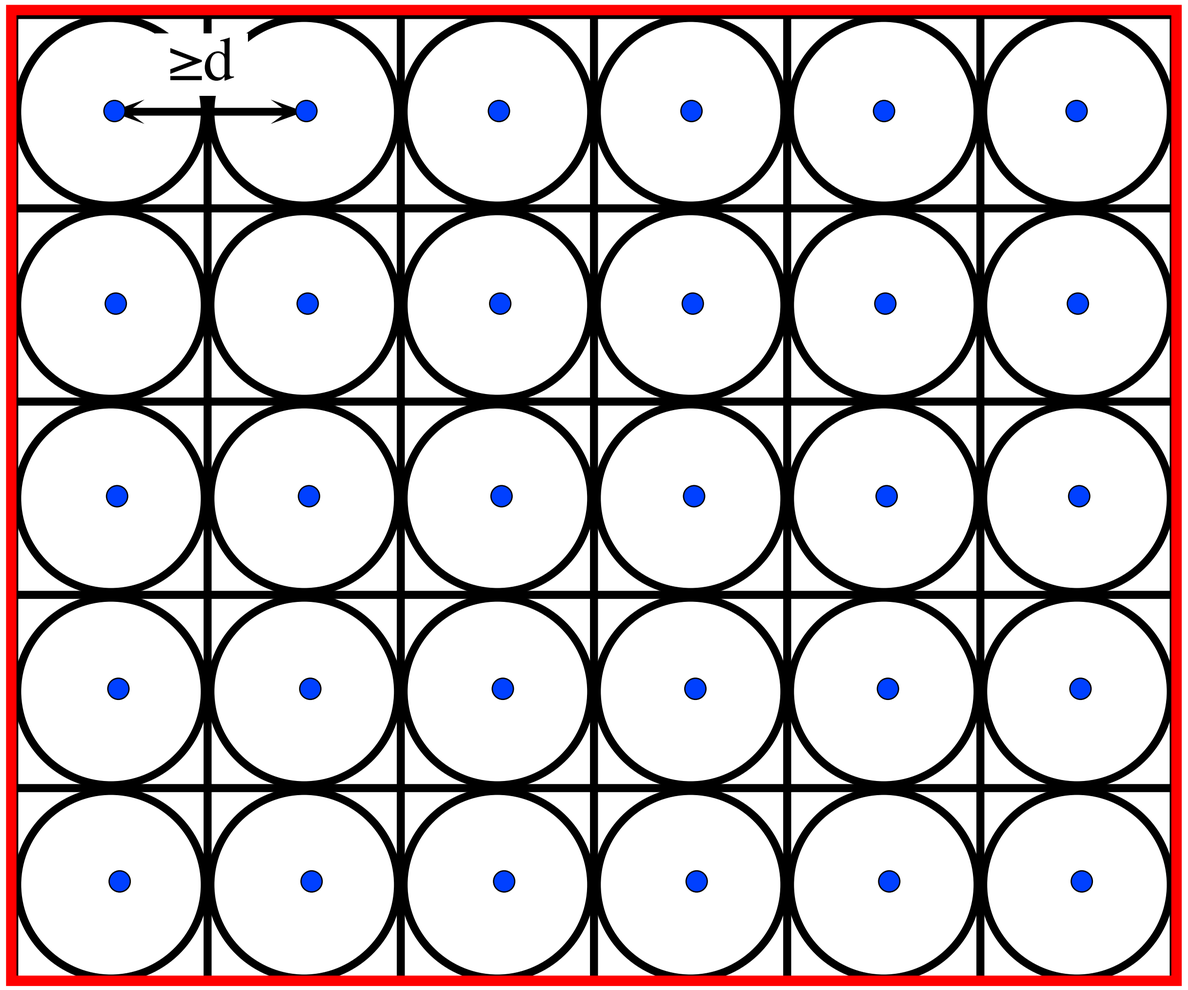
D. Gottesman and A. Smith

(0)

Classical Error Correcting Codes

(classical) error-correcting codes

q^n (words)



q^k (codewords)

[n,k,d] linear error-correcting code
length n, dimension k,
corrects d-1 erasures, (d-1)/2 errors

(1)

Quantum Error Correcting Codes

Q: (over GF(3))

$$|0\rangle \rightarrow |000\rangle + |111\rangle + |222\rangle$$
$$|1\rangle \rightarrow |012\rangle + |120\rangle + |201\rangle$$
$$|2\rangle \rightarrow |021\rangle + |102\rangle + |210\rangle$$

$$\mathbf{Q}|\psi\rangle = H_1 \otimes H_2 \otimes H_3$$

Q=[[3,1,2]] corrects 2-1=1 erasure.

$$|0\rangle \otimes H_2 \otimes H_3 \rightarrow (-H_2 - H_3 \bmod 3) \otimes H_2 \otimes H_3$$
$$H_1 \otimes |0\rangle \otimes H_3 \rightarrow H_1 \otimes (-H_3 - H_1 \bmod 3) \otimes H_3$$
$$H_1 \otimes H_2 \otimes |0\rangle \rightarrow H_1 \otimes H_2 \otimes (-H_1 - H_2 \bmod 3)$$

Calderbank-Shor-Steane Q-ECCs

Let C_1, C_2 be two linear codes such that

$$\{0\}\subseteq C_2\subseteq C_1\subseteq \mathbf{F}^n \qquad \{0\}\subseteq C_1^{\perp}\subseteq C_2^{\perp}\subseteq \mathbf{F}^n$$

For $v\in C_1$ define

$$v\rightarrow \frac{1}{\sqrt{|C_2|}}\sum_{w\in C_2}|v+w\rangle$$

$$Q=\left\{\frac{1}{\sqrt{|C_2|}}\sum_{w\in C_2}|w+v\rangle : v\in C_1\right\}$$

For $v\in C_2^{\perp}$ define

$$v\rightarrow \frac{1}{\sqrt{|C_1^{\perp}|}}\sum_{w\in C_1^{\perp}}|v+w\rangle$$

$$Q^{*}=\left\{\frac{1}{\sqrt{|C_1^{\perp}|}}\sum_{w\in C_1^{\perp}}|w+v\rangle : v\in C_2^{\perp}\right\}$$

CSS Q-ECCs

Let $C_1=[n,k_1,d_1]$, $C_2^{\perp}=[n,n-k_2,d_2]$ be two linear codes

$$\begin{aligned}\dim(Q) &= \dim(C_1)\text{-}\dim(C_2^{\perp}) \\ &= k_1-k_2 \\ &= \dim(C_2^{\perp})\text{-}\dim(C_1) = \dim(Q^*)\end{aligned}$$

$$\mathrm{d}(Q) = \mathrm{d}(Q^*) = \min\{\mathrm{d}(C_1),\mathrm{d}(C_2^{\perp})\} = \min\{d_1,d_2\}$$

$$Q = [[\ n,\ k_1-k_2,\ \min\{d_1,d_2\}\ ]] = Q^*$$

CSS Q-ECCs

EXAMPLE: Quantum Reed-Solomon codes (Aharonov-BenOr)

Let $q=4t$

$C_1 = [4t,2t+1,2t]$ ERS-code over GF(q)
$C_2 = [4t,2t,2t+1]$ ERS-code over GF(q)

$\dim(Q) = \dim(Q^*) = 1$
$\mathrm{d}(Q) = \mathrm{d}(Q^*) = 2t$

$Q,Q^*= [[4t, 1, 2t]]$ QRS-code over GF(q)

$Q,Q^*= [[n, 1, n/2]]$ QRS-code over GF(q), $q=n$

Theorem: No QECC tolerates $t \geq n/4$

Proof:

- No cloning says that no QECC can correct $n/2$ erasures
- Fact: Any QECC which corrects t errors can correct $2t$ erasures and conversely
- Thus no QECC tolerates $n/4$ errors
- All these arguments work ***regardless of the size*** of the components of QECC (size of the field of definition)

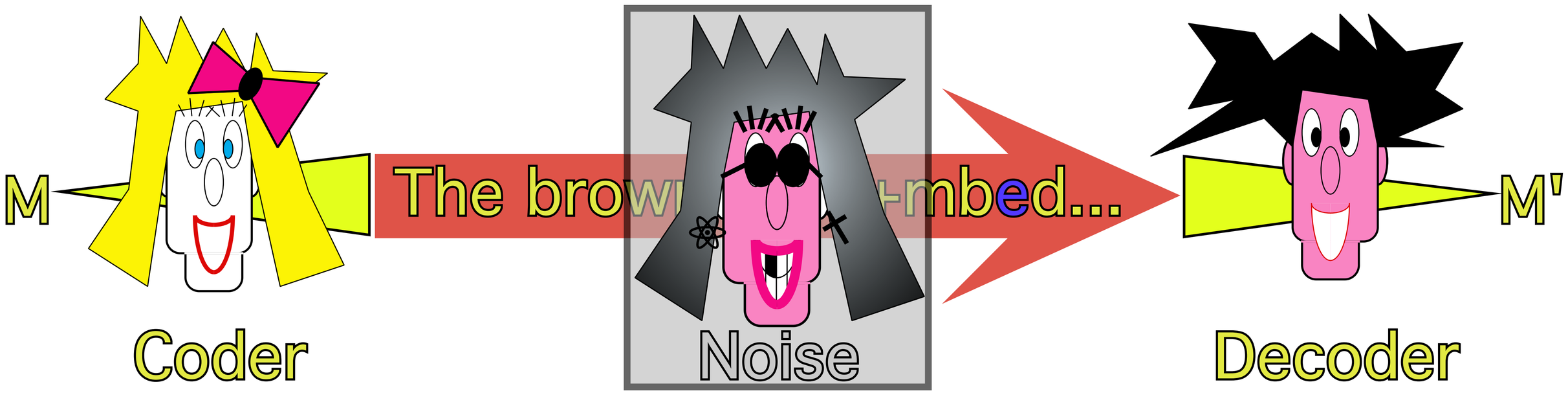
- **Fact: Any QECC which corrects t errors can correct $2t$ erasures and conversely**

If small error probability is acceptable

Error probability of correcting

is taken over choices* of code
but
NOT over distribution of errors

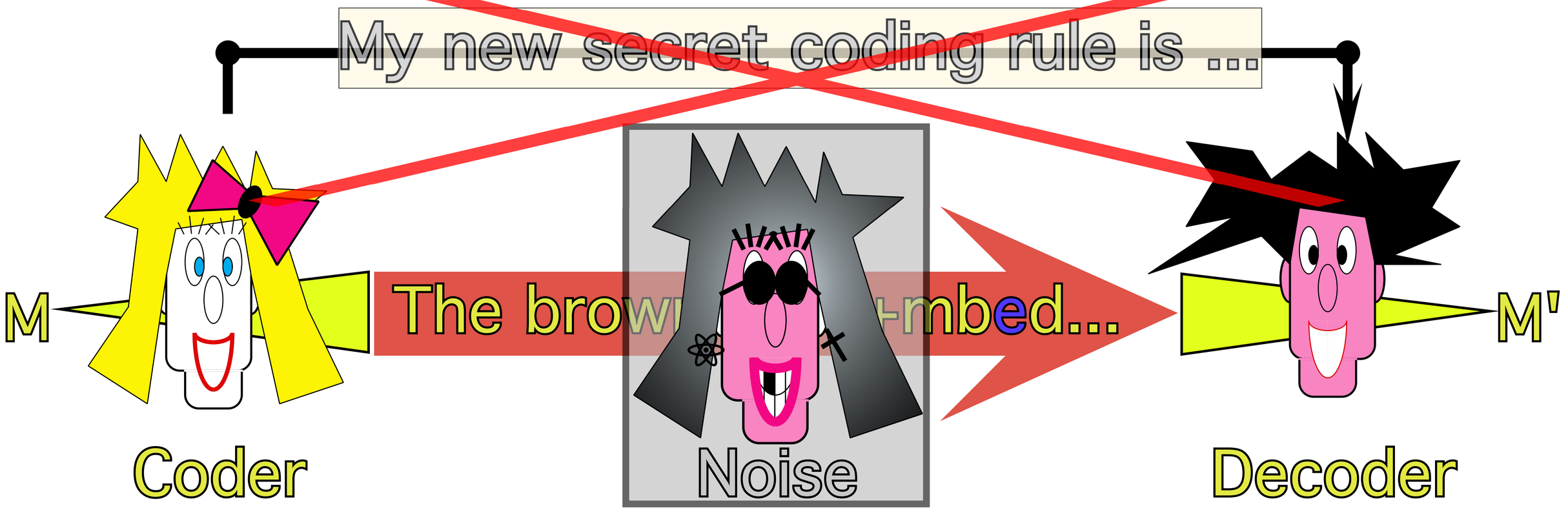
*All communications MUST go through the noisy channel.



Error probability of correcting

is taken over choices* of code
but
NOT over distribution of errors

*All communications MUST go through the noisy channel.
We disallow private channel between coder and decoder.



(4)

Quantum Codes Correcting* 1 Arbitrary Error out of 3 positions

*except with exponentially small probability

Q: (over GF(3))

$$|0\rangle \rightarrow |000\rangle + |111\rangle + |222\rangle$$
$$|1\rangle \rightarrow |012\rangle + |120\rangle + |201\rangle$$
$$|2\rangle \rightarrow |021\rangle + |102\rangle + |210\rangle$$

Q=[[3,1,2]] corrects one erasure.

$$\mathbf{Q}|\psi\rangle = H_1 \otimes H_2 \otimes H_3$$

$\mathcal{Q}$: (over GF(q), q»3)

$$\mathcal{H}_1=\langle A_{K_1}(H_1),K_2,K_3\rangle$$
$$\mathcal{H}_2=\langle A_{K_2}(H_2),K_3,K_1\rangle$$
$$\mathcal{H}_3=\langle A_{K_3}(H_3),K_1,K_2\rangle$$

$\mathcal{Q}$=[[3,1,2]] correcting one arbitrary error!

$$\mathcal{Q}|\psi\rangle=\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_3$$

If zero/one error occurred (case 1)

but all keys K_1, K_2, K_3 **agree in** $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$

CASE 1)

$$\mathcal{H}_1 = \langle A_{K_1}(H_1), K_2, K_3 \rangle$$
$$\mathcal{H}_2 = \langle A_{K_2}(H_2), K_3, K_1 \rangle$$
$$\mathcal{H}_3 = \langle A_{K_3}(H_3), K_1, K_2 \rangle$$

If zero/one error occurred (case 1)

but all keys K_1, K_2, K_3 **agree in** $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$

CASE 1)

$$\mathcal{H}_1 = \langle A_{K_1}(H_1), K_2, K_3 \rangle$$
$$\mathcal{H}_2 = \langle A_{K_2}(H_2), K_3, K_1 \rangle$$
$$\mathcal{H}_3 = \langle A_{K_3}(H_3), K_1, K_2 \rangle$$

- **using keys** K_1, K_2, K_3

try to get H_1 **from** $\mathcal{H}_1$, H_2 **from** $\mathcal{H}_2$, H_3 **from** $\mathcal{H}_3$

If zero/one error occurred (case 1)

but all keys K_1,K_2,K_3 **agree in** $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3$

CASE 1)

$$\mathcal{H}_1=\langle A_{K_1}(H_1),K_2,K_3\rangle$$
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- **using keys** K_1,K_2,K_3

try to get H_1 **from** $\mathcal{H}_1$, H_2 **from** $\mathcal{H}_2$, H_3 **from** $\mathcal{H}_3$

- **at most one quantum authentication may fail**

If zero/one error occurred (case 1) √

but all keys K_1,K_2,K_3 agree in $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3$

CASE 1)

$$\mathcal{H}_1=\langle A_{K_1}(H_1),K_2,K_3\rangle$$
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- using keys K_1,K_2,K_3

try to get H_1 from $\mathcal{H}_1$, H_2 from $\mathcal{H}_2$, H_3 from $\mathcal{H}_3$

- at most one quantum authentication may fail

- using Q's algorithm for fixing one erasure get

$|\psi\rangle$ from $|0\rangle\otimes H_2\otimes H_3$, $H_1\otimes|0\rangle\otimes H_3$ or $H_1\otimes H_2\otimes|0\rangle$

CASE 1) √

$$\mathcal{H}_1 = \langle A_{K_1}(H_1), K_2, K_3 \rangle$$
$$\mathcal{H}_2 = \langle A_{K_2}(H_2), K_3, K_1 \rangle$$
$$\mathcal{H}_3 = \langle A_{K_3}(H_3), K_1, K_2 \rangle$$

CASE 2)

$$\mathcal{H}_a = \langle A_{K_a}(H_a), K_i, K_b \rangle$$
$$\mathcal{H}_i = \langle A_{K_i}(H_i), K_b, K_a \rangle$$
$$\mathcal{H}_b = \langle A_{K_b}(H_b), K_a, K_i \rangle$$

CASE 3)

$$\mathcal{H}_a = \langle A_{K_a}(H_a), K_i, K_b \rangle$$
$$\mathcal{H}_i = \langle A_{K_i}(H_i), K_b, K_a \rangle$$
$$\mathcal{H}_b = \langle A_{K_b}(H_b), K_a, K_i \rangle$$

If one error occurred (case 2)

but for some ***i*** **the keys** K_a, K_b, $a \neq i \neq b$

disagree in $\mathcal{H}_b$ **vs** $\mathcal{H}_i$, **and in** $\mathcal{H}_a$ **vs** $\mathcal{H}_i$

($\mathcal{H}_i$ **must be wrong)**

CASE 2)

$$\mathcal{H}_a = \langle A_{K_a}(H_a), K_i, K_b \rangle$$
$$\mathcal{H}_i = \langle A_{K_i}(H_i), K_b, K_a \rangle$$
$$\mathcal{H}_b = \langle A_{K_b}(H_b), K_a, K_i \rangle$$

If one error occurred (case 2)

but for some i **the keys** K_a, K_b, $a \neq i \neq b$

disagree in $\mathcal{H}_b$ **vs** $\mathcal{H}_i$, **and in** $\mathcal{H}_a$ **vs** $\mathcal{H}_i$

($\mathcal{H}_i$ **must be wrong**)

- **using keys** K_a **in** $\mathcal{H}_b$, **and** K_b **in** $\mathcal{H}_a$

get H_a **from** $\mathcal{H}_a$, H_b **from** $\mathcal{H}_b$

CASE 2)

$$\mathcal{H}_a = \langle A_{K_a}(H_a), K_i, K_b \rangle$$
$$\mathcal{H}_i = \langle A_{K_i}(H_i), K_b, K_a \rangle$$
$$\mathcal{H}_b = \langle A_{K_b}(H_b), K_a, K_i \rangle$$

If one error occurred (case 2)

but for some *i* the keys K_a,K_b, $a \neq i \neq b$

disagree in $\mathcal{H}_b$ vs $\mathcal{H}_i$, and in $\mathcal{H}_a$ vs $\mathcal{H}_i$

($\mathcal{H}_i$ must be wrong)

- **using keys K_a in $\mathcal{H}_b$, and K_b in $\mathcal{H}_a$**

  get H_a from $\mathcal{H}_a$, H_b from $\mathcal{H}_b$

- **no Q-authentication may fail since error at *i***

If one error occurred (case 2) √

but for some ***i*** **the keys** K_a, K_b, $a \neq i \neq b$

disagree in $\mathcal{H}_b$ **vs** $\mathcal{H}_i$, **and in** $\mathcal{H}_a$ **vs** $\mathcal{H}_i$

($\mathcal{H}_i$ **must be wrong)**

- **using keys** K_a **in** $\mathcal{H}_b$, **and** K_b **in** $\mathcal{H}_a$

  get H_a **from** $\mathcal{H}_a$, H_b **from** $\mathcal{H}_b$

- **no Q-authentication may fail since error at** ***i***

- **using Q's algorithm for fixing one erasure get**

  $|\psi\rangle$ **from** $|0\rangle \otimes H_2 \otimes H_3$, $H_1 \otimes |0\rangle \otimes H_3$ **or** $H_1 \otimes H_2 \otimes |0\rangle$

If one error occurred (case 3)

but for some *i* only key K_i disagree in

$\mathcal{H}_a$ vs $\mathcal{H}_b$, $a \neq i \neq b$. ($\mathcal{H}_i$ must be right)

CASE 3)

$$\mathcal{H}_a = \langle A_{K_a}(H_a), K_i, K_b \rangle$$

$$\mathcal{H}_i = \langle A_{K_i}(H_i), K_b, K_a \rangle$$

$$\mathcal{H}_b = \langle A_{K_b}(H_b), K_a, K_i \rangle$$

If one error occurred (case 3)

but for some *i* only key K_i disagree in

$\mathcal{H}_a$ vs $\mathcal{H}_b$, $a \neq i \neq b$. ($\mathcal{H}_i$ must be right)

- **using keys K_a in $\mathcal{H}_i$, and K_b in $\mathcal{H}_i$**

try to get H_a from $\mathcal{H}_a$, H_b from $\mathcal{H}_b$

CASE 3)

$$\mathcal{H}_a = \langle A_{K_a}(H_a), K_i, K_b \rangle$$
$$\mathcal{H}_i = \langle A_{K_i}(H_i), K_b, K_a \rangle$$
$$\mathcal{H}_b = \langle A_{K_b}(H_b), K_a, K_i \rangle$$

If one error occurred (case 3)

but for some *i* only key K_i disagree in

$\mathcal{H}_a$ vs $\mathcal{H}_b$, $a \neq i \neq b$. ($\mathcal{H}_i$ must be right)

- **using keys K_a in $\mathcal{H}_i$, and K_b in $\mathcal{H}_i$**

try to get H_a from $\mathcal{H}_a$, H_b from $\mathcal{H}_b$

- **at most one quantum authentication may fail**

CASE 3)

$$\mathcal{H}_a = \langle A_{K_a}(H_a), K_i, K_b \rangle$$
$$\mathcal{H}_i = \langle A_{K_i}(H_i), K_b, K_a \rangle$$
$$\mathcal{H}_b = \langle A_{K_b}(H_b), K_a, K_i \rangle$$

If one error occurred (case 3)

but for some *i* only key K_i disagree in

$\mathcal{H}_a$ vs $\mathcal{H}_b$, *a≠i≠b*. ($\mathcal{H}_i$ must be right)

- **using keys K_a in $\mathcal{H}_i$, and K_b in $\mathcal{H}_i$**

try to get H_a from $\mathcal{H}_a$, H_b from $\mathcal{H}_b$

- **at most one quantum authentication may fail: if authentication fails at *a* (or *b*)**

  then use key K_i in $\mathcal{H}_b$ ($\mathcal{H}_a$), and get H_i from $\mathcal{H}_i$

If one error occurred (case 3) √

but for some *i* only key K_i disagree in

$\mathcal{H}_a$ vs $\mathcal{H}_b$, *a≠i≠b*. ($\mathcal{H}_i$ must be right)

- **using keys K_a in $\mathcal{H}_i$, and K_b in $\mathcal{H}_i$**

try to get H_a from $\mathcal{H}_a$, H_b from $\mathcal{H}_b$

- **at most one quantum authentication may fail: if authentication fails at *a* (or *b*)**

  then use key K_i in $\mathcal{H}_b$ ($\mathcal{H}_a$), and get H_i from $\mathcal{H}_i$

- **using Q's algorithm for fixing one erasure get**

$|\psi\rangle$ from $|0\rangle\otimes H_2\otimes H_3$, $H_1\otimes|0\rangle\otimes H_3$ or $H_1\otimes H_2\otimes|0\rangle$

(5)

Classical Secret Sharing

Classical Secret Sharing

$SS_{n,t}[K]$ = **set of n-tuples of values s.t.**

- **any ≤t-1 values = no info about K**
- **any ≥t values = full info about K.**

$$SS_{n,t}[K] =$$

$$\{ \langle p(\omega_1), p(\omega_2), ..., p(\omega_n) \rangle \mid$$

$$p(x) = a_{t-1}x^{t-1} + a_{t-2}x^{t-2} + ... + a_1 x + K,$$

$$a_{t-1}, a_{t-2}, ..., a_1 \in GF(q), q \geq n \}$$

(6)

Quantum Codes Correcting* up to (n-1)/2 Arbitrary Errors out of n positions

*except with exponentially small probability

Ingredients

Quantum Authentication Scheme:

$$|\psi\rangle,K \rightarrow A_K(|\psi\rangle)$$

Classical Authentication Scheme:

$$m,\kappa \rightarrow (m,\alpha_\kappa(m))$$

Classical Secret Sharing Scheme:

$$\langle s_1,s_2,\ldots,s_n\rangle \in_R SS_{n,t}[X]$$

$\mathcal{Q}$: (over GF(q), q»3)

$$\mathcal{H}_1=\langle A_{K_1}(H_1),\mathbf{s}_1,\alpha_{\kappa_{21}}(\mathbf{s}_1),\alpha_{\kappa_{31}}(\mathbf{s}_1),\kappa_{12},\kappa_{13}\rangle$$
$$\mathcal{H}_2=\langle A_{K_2}(H_2),\mathbf{s}_2,\alpha_{\kappa_{32}}(\mathbf{s}_2),\alpha_{\kappa_{12}}(\mathbf{s}_2),\kappa_{23},\kappa_{21}\rangle$$
$$\mathcal{H}_3=\langle A_{K_3}(H_3),\mathbf{s}_3,\alpha_{\kappa_{13}}(\mathbf{s}_3),\alpha_{\kappa_{23}}(\mathbf{s}_3),\kappa_{31},\kappa_{32}\rangle$$

$\mathcal{Q}$=[[3,1,2]] correcting one arbitrary error!

$\mathcal{Q}|\psi\rangle=\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_3$

$$\langle \mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3\rangle \in_{\mathbf{R}} \mathbf{SS}_{3,2}[\mathbf{K}_1:\mathbf{K}_2:\mathbf{K}_3]$$

def: s_i is <u>valid</u> if at most ONE classical authentication of it fails.

$$\mathcal{H}_1=\langle A_{K_1}(H_1),\mathbf{s}_1,\alpha_{\kappa_{21}}(\mathbf{s}_1),\alpha_{\kappa_{31}}(\mathbf{s}_1),\kappa_{12},\kappa_{13}\rangle$$
$$\mathcal{H}_2=\langle A_{K_2}(H_2),\mathbf{s}_2,\alpha_{\kappa_{32}}(\mathbf{s}_2),\alpha_{\kappa_{12}}(\mathbf{s}_2),\kappa_{23},\kappa_{21}\rangle$$
$$\mathcal{H}_3=\langle A_{K_3}(H_3),\mathbf{s}_3,\alpha_{\kappa_{13}}(\mathbf{s}_3),\alpha_{\kappa_{23}}(\mathbf{s}_3),\kappa_{31},\kappa_{32}\rangle$$

def: s_i is valid if at most ONE classical authentication of it fails.

claim: #{ i | s_i is valid } ≥ 2

$$\mathcal{H}_1=\langle A_{K_1}(H_1),s_1,\alpha_{\kappa_{21}}(s_1),\alpha_{\kappa_{31}}(s_1),\kappa_{12},\kappa_{13}\rangle$$
$$\mathcal{H}_2=\langle A_{K_2}(H_2),s_2,\alpha_{\kappa_{32}}(s_2),\alpha_{\kappa_{12}}(s_2),\kappa_{23},\kappa_{21}\rangle$$
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def: s_i is valid if at most ONE classical authentication of it fails.

claim: #{ i | s_i is valid } ≥ 2

$[K_1:K_2:K_3]$ is recovered from { s_i is valid }

$$\mathcal{H}_1=\langle A_{K_1}(H_1),s_1,\alpha_{\kappa_{21}}(s_1),\alpha_{\kappa_{31}}(s_1),\kappa_{12},\kappa_{13}\rangle$$
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def: S_i is valid if at most ONE classical authentication of it fails.

claim: #{ i | S_i is valid } ≥ 2

$[K_1:K_2:K_3]$ is recovered from { S_i is valid }

- **using keys K_1, K_2, K_3**

try to get H_1 from $\mathcal{H}_1$, H_2 from $\mathcal{H}_2$, H_3 from $\mathcal{H}_3$

- **at most one quantum authentication may fail**

- **using Q's algorithm for fixing one erasure get**

$|\psi\rangle$ from $|0\rangle \otimes H_2 \otimes H_3$, $H_1 \otimes |0\rangle \otimes H_3$ or $H_1 \otimes H_2 \otimes |0\rangle$

Generalization

Q: (over GF(q))

Q=[[n,k,d]] corrects d-1<n/2 erasures

$$Q|\psi\rangle = H_1 \otimes H_2 \otimes H_3 \otimes \ldots \otimes H_n$$

$\mathcal{Q}$: (over GF(q'), q'»q)

$\mathcal{H}_1,\ldots,\mathcal{H}_i,\ldots,\mathcal{H}_n$

$$\mathcal{H}_i = \langle A_{K_i}(H_i), s_i, \alpha_{K_{1i}}(s_i),\ldots,\alpha_{K_{(i-1)i}}(s_i),\alpha_{K_{(i+1)i}}(s_i),\ldots,\alpha_{K_{ni}}(s_i), K_{i1},\ldots,K_{i(i-1)},K_{i(i+1)},\ldots,K_{in}\rangle$$

$\mathcal{Q}$=[[n,k,d]] correcting d-1 arbitrary errors!

$$\mathcal{Q}|\psi\rangle = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$$

$$\langle s_1, s_2, \ldots, s_n \rangle \in {}_R SS_{n,n-d}[K_1 : K_2 : K_3 : \ldots : K_n]$$

def: S_i is valid if at most *d*-1 classical authentication of it fails.

claim: #{ i | S_i is valid } ≥ *n*-*d*+1 ≥ *n*/2

[K_1:K_2:K_3:…:K_n] is recovered from { S_i is valid }

- **using keys $K_1, K_2, K_3, \cdots, K_n$ try to get each H_i from $\mathcal{H}_i$**
- **at most *d*-1 quantum authentications may fail**
- **using Q's algorithm for fixing *d*-1 erasures get $|\psi\rangle$ from $H_1 \otimes H_2 \otimes \ldots \otimes H_n$, where *d*-1 parts are $|0\rangle$.**

Further Applications and Open Problems

- **Achieving classical bounds for VQSS and MPQC (Crépeau,Gottesman,Smith)**

- **Length n QECC correcting d < n/2 arbitrary errors (with exponentially small probability)**

  - **More natural constructions**

  - **Constructions over smaller fields**

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