An Introduction to Quantum Error Correction

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Quantum computers have a continuum of states, so it would seem, at first glance, that they cannot do this. For instance, a likely source of error is over-rotation: a state $\alpha|0\rangle + \beta|1\rangle$ might be supposed to become $\alpha|0\rangle + \beta e^{i\phi}|1\rangle$, but instead becomes $\alpha|0\rangle + \beta e^{i(\phi+\delta)}|1\rangle$. The actual state is very close to the correct state, but it is still wrong. If we don't do something about this, the small errors will build up over the course of the computation, and eventually will become a big error.

Furthermore, quantum states are intrinsically delicate: looking at one collapses it. $\alpha|0\rangle + \beta|1\rangle$ becomes $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$. The environment is constantly trying to look at the state, a process called *decoherence*. One goal of quantum error correction will be to prevent the environment from looking at the data.

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There is a well-developed theory of classical error-correcting codes, but it doesn't apply here, at least not directly. For one thing, we need to keep the phase correct as well as correcting bit flips. There is another problem, too. Consider the simplest classical code, the repetition code:

$$0 \to 000 \tag{1}$$

$$1 \to 111 \tag{2}$$

It will correct a state such as 010 to the majority value (becoming 000 in this case).[1]

We might try a quantum repetition code:

$$|\psi\rangle \to |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \tag{3}$$

However, no such code exists because of the No-Cloning theorem [**6, 20**]:

THEOREM 1 (No-Cloning). *There is no quantum operation that takes a state* $|\psi\rangle$ *to* $|\psi\rangle \otimes |\psi\rangle$ *for all states* $|\psi\rangle$.

PROOF. This fact is a simple consequence of the linearity of quantum mechanics. Suppose we had such an operation and $|\psi\rangle$ and $|\phi\rangle$ are distinct. Then, by the definition of the operation,

$$|\psi\rangle \to |\psi\rangle|\psi\rangle \tag{4}$$

$$|\phi\rangle \to |\phi\rangle|\phi\rangle \tag{5}$$

$$|\psi\rangle + |\phi\rangle \to (|\psi\rangle + |\phi\rangle)\,(|\psi\rangle + |\phi\rangle)\,. \tag{6}$$

(Here, and frequently below, I omit normalization, which is generally unimportant.)

But by linearity,

$$|\psi\rangle + |\phi\rangle \to |\psi\rangle|\psi\rangle + |\phi\rangle|\phi\rangle. \tag{7}$$

This differs from (6) by the crossterm

$$|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle. \tag{8}$$

□

2. The nine-qubit code

$$|0\rangle \to |\overline{0}\rangle = (|000\rangle + |111\rangle)\,(|000\rangle + |111\rangle)\,(|000\rangle + |111\rangle) \tag{9}$$

$$|1\rangle \to |\overline{1}\rangle = (|000\rangle - |111\rangle)\,(|000\rangle - |111\rangle)\,(|000\rangle - |111\rangle) \tag{10}$$

Note that this does not violate the No-Cloning theorem, since an arbitrary codeword will be a linear superposition of these two states

$$\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle \neq [\alpha(|000\rangle + |111\rangle) + \beta(|000\rangle - |111\rangle)]^{\otimes 3}\,. \tag{11}$$

The superposition is linear in α and β. The complete set of codewords for this (or any other) quantum code form a linear subspace of the Hilbert space, the *coding space*.

The inner layer of this code corrects bit flip errors: We take the majority within each set of three, so

$$|010\rangle \pm |101\rangle \to |000\rangle \pm |111\rangle. \tag{12}$$

The outer layer corrects phase flip errors: We take the majority of the three signs, so

$$(|\cdot\rangle + |\cdot\rangle)(|\cdot\rangle - |\cdot\rangle)(|\cdot\rangle + |\cdot\rangle) \to (|\cdot\rangle + |\cdot\rangle)(|\cdot\rangle + |\cdot\rangle)(|\cdot\rangle + |\cdot\rangle). \tag{13}$$

The bit flip, phase flip, and combined bit and phase flip errors are important, so let's take a short digression to discuss them.

Identity	$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$I\|a\rangle = \|a\rangle$
Bit Flip	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$X\|a\rangle = \|a \oplus 1\rangle$
Phase Flip	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$Z\|a\rangle = (-1)^a\|a\rangle$
Bit & Phase	$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iXZ$	$Y\|a\rangle = i(-1)^a\|a \oplus 1\rangle$

TABLE 1. The Pauli matrices

The group generated by tensor products of these 4 operators is called the Pauli group. X, Y, and Z anticommute: $XZ = -ZX$ (also written $\{X, Z\} = 0$). Similarly, $\{X, Y\} = 0$ and $\{Y, Z\} = 0$. Thus, the n-qubit Pauli group $\mathcal{P}_n$ consists of the 4^n tensor products of I, X, Y, and Z, and an overall phase of ± 1 or $\pm i$, for a total of 4^{n+1} elements. The phase of the operators used is not generally very important, but we can't discard it completely. For one thing, the fact that this is not an Abelian group is quite important, and we would lose that if we dropped the phase!

$\mathcal{P}_n$ is useful because of its nice algebraic properties. Any pair of elements of $\mathcal{P}_n$ either commute or anticommute. Also, the square of any element of $\mathcal{P}_n$ is ± 1. We shall only need to work with the elements with square $+1$, which are tensor products of I, X, Y, and Z with an overall sign ± 1; the phase i is only necessary to make $\mathcal{P}_n$ a group. Define the *weight* of an operator in $\mathcal{P}_n$ to be the number of tensor factors which are not I. Thus, $X \otimes Y \otimes I$ has weight 2.

Another reason the Pauli matrices are important is that they span the space of 2×2 matrices, and the n-qubit Pauli group spans the space of $2^n \times 2^n$ matrices. For instance, if we have a general phase error

$$R_{\theta/2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = e^{i\theta/2} \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{14}$$

(again, the overall phase does not matter), we can write it as

$$R_{\theta/2} = \cos\frac{\theta}{2}\, I - i \sin\frac{\theta}{2}\, Z. \tag{15}$$

It turns out that our earlier error correction procedure will also correct this error, without any additional effort.

For instance, the earlier procedure might use some extra qubits (*ancilla* qubits) that are initialized to $|0\rangle$ and record what type of error occurred. Then we look at the ancilla and invert the error it tells us:

$$Z\left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |0\rangle_{\text{anc}} \to Z\left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |Z\rangle_{\text{anc}} \tag{16}$$

$$\to \left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |Z\rangle_{\text{anc}} \tag{17}$$

$$I\left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |0\rangle_{\text{anc}} \to I\left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |\text{no error}\rangle_{\text{anc}} \tag{18}$$

$$\to \left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |\text{no error}\rangle_{\text{anc}} \tag{19}$$

When the actual error is $R_{\theta/2}$, recording the error in the ancilla gives us a superposition:

$$\cos\frac{\theta}{2}\, I\left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |\text{no error}\rangle_{\text{anc}} - i\sin\frac{\theta}{2}\, Z\left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |Z\rangle_{\text{anc}} \tag{20}$$

Then we measure the ancilla, which with probability $\sin^2\theta/2$ gives us

$$Z\left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |Z\rangle_{\text{anc}}, \tag{21}$$

and with probability $\cos^2\theta/2$ gives us

$$I\left(\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle\right) \otimes |\text{no error}\rangle_{\text{anc}}. \tag{22}$$

In each case, inverting the error indicated in the ancilla restores the original state.

THEOREM 2. *If a quantum code corrects errors A and B, it also corrects any linear combination of A and B. In particular, if it corrects all weight t Pauli errors, then the code corrects all t-qubit errors.*

So far, we have only considered individual unitary errors that occur on the code. But we can easily add in all possible quantum errors. The most general quantum operation, including decoherence, interacts the quantum state with some extra qubits via a unitary operation, then discards some qubits. This process can turn pure quantum states into mixed quantum states, which are normally described using density matrices. We can write the most general operation as a transformation on density matrices

$$\rho \to \sum_i E_i \rho E_i^\dagger, \tag{23}$$

where the E_is are normalized so $\sum E_i^\dagger E_i = I$. The density matrix ρ can be considered to represent an ensemble of pure quantum states $|\psi\rangle$, each of which, in this case, should be in the coding space of the code. Then this operation simply performs the following operation on each $|\psi\rangle$:

$$|\psi\rangle \to E_i|\psi\rangle \text{ with probability } |E_i|\psi\rangle|^2. \tag{24}$$

If we can correct each of the individual errors E_i, then we can correct this general error as well. For instance, for quantum operations that only affect a single qubit of the code, E_i will necessarily be in the linear span of I, X, Y, and Z, so we can correct it. Thus, in the statement of theorem 2, "all t-qubit errors" really does apply to *all* t-qubit errors, not just unitary ones.

3. General properties of quantum error-correcting codes

One useful feature was *linearity*, which will be true of any quantum code. We only need to correct a basis of errors (I, X, Y, and Z in the one-qubit case), and all other errors will follow, as per theorem 2.

In any code, we must never confuse $|\overline{0}\rangle$ with $|\overline{1}\rangle$, even in the presence of errors. That is, $E|\overline{0}\rangle$ is orthogonal to $F|\overline{1}\rangle$:

$$\langle\overline{0}|E^\dagger F|\overline{1}\rangle = 0. \tag{26}$$

It is *sufficient* to distinguish error E from error F when they act on $|\overline{0}\rangle$ and $|\overline{1}\rangle$. Then a measurement will tell us exactly what the error is and we can correct it:

$$\langle\overline{0}|E^\dagger F|\overline{0}\rangle = \langle\overline{1}|E^\dagger F|\overline{1}\rangle = 0 \tag{27}$$

for $E \neq F$.

But (27) is not *necessary*: in the nine-qubit code, we cannot distinguish between Z_1 and Z_2, but that is OK, since we can correct either one with a single operation. To understand the necessary condition, it is helpful to look at the operators $F_1 = (Z_1 + Z_2)/2$ and $F_2 = (Z_1 - Z_2)/2$ instead of Z_1 and Z_2. F_1 and F_2 span the same space as Z_1 and Z_2, so Shor's code certainly corrects them; let us try to understand how. When we use the Fs as the basis errors, now equation (27) *is* satisfied. That means we can make a measurement and learn what the error is. We also have to invert it, and this is a potential problem, since F_1 and F_2 are not unitary. However, F_1 acts the same way as Z_1 on the coding space, so $Z_1^\dagger$ suffices to invert F_1 on the states of interest. F_2 acts the same way as the 0 operator on the coding space. We can't invert this, but we don't need to — since F_2 annihilates codewords, it can never contribute a component to the actual state of the system.

The requirement to invert the errors produces a third condition:

$$\langle\overline{0}|E^\dagger E|\overline{0}\rangle = \langle\overline{1}|E^\dagger E|\overline{1}\rangle. \tag{28}$$

Either this value is nonzero, as for F_1, in which case some unitary operator will act the same way as E on the coding space, or it will be zero, as for F_2, in which case E annihilates codewords and never arises.

THEOREM 3. *Suppose $\mathcal{E}$ is a linear space of errors acting on the Hilbert space $\mathcal{H}$. Then a subspace C of $\mathcal{H}$ forms a quantum error-correcting code correcting the errors $\mathcal{E}$ iff*

$$\langle\psi|E^\dagger E|\psi\rangle = C(E) \tag{29}$$

for all $E \in \mathcal{E}$. The function $C(E)$ does not depend on the state $|\psi\rangle$.

PROOF. Suppose $\{E_a\}$ is a basis for $\mathcal{E}$ and $\{|\psi_i\rangle\}$ is a basis for C. By setting E and $|\psi\rangle$ equal to the basis elements and to the sum and difference of two basis elements (with or without a phase factor i), we can see that (29) is equivalent to

$$\langle\psi_i|E_a^\dagger E_b|\psi_j\rangle = C_{ab}\delta_{ij}, \tag{30}$$

where C_{ab} is a Hermitian matrix independent of i and j.

Suppose equation (30) holds. We can diagonalize C_{ab}. This involves choosing a new basis $\{F_a\}$ for $\mathcal{E}$, and the result is equations (26), (27), and (28). The arguments before the theorem show that we can measure the error, determine it uniquely (in the new basis), and invert it (on the coding space). Thus, we have a quantum error-correcting code.

Now suppose we have a quantum error-correcting code, and let $|\psi\rangle$ and $|\phi\rangle$ be two distinct codewords. Then we must have

$$\langle\psi|E^\dagger E|\psi\rangle = \langle\phi|E^\dagger E|\phi\rangle \tag{31}$$

for all E. That is, (29) must hold. If not, E changes the relative size of $|\psi\rangle$ and $|\phi\rangle$. Both $|\psi\rangle + |\phi\rangle$ and $|\psi\rangle + c|\phi\rangle$ are valid codewords, and

$$E(|\psi\rangle + |\phi\rangle) = N(|\psi\rangle + c|\phi\rangle), \tag{32}$$

where N is a normalization factor and

$$c = \langle\psi|E^\dagger E|\psi\rangle / \langle\phi|E^\dagger E|\phi\rangle. \tag{33}$$

The error E will actually change the encoded state, which is a failure of the code, unless $c = 1$.

□

There is a slight subtlety to the phrasing of equation (29). We require $\mathcal{E}$ to be a linear space of errors, which means that it must be closed under sums of errors which may act on different qubits. In contrast, for a code that corrects t errors, in (30), it is safe to consider only E_a and E_b acting on just t qubits. We can restrict even further, and only use Pauli operators as E_a and E_b, since they will span the space of t-qubit errors. This leads us to a third variation of the condition:

$$\langle\psi|E|\psi\rangle = C'(E), \tag{34}$$

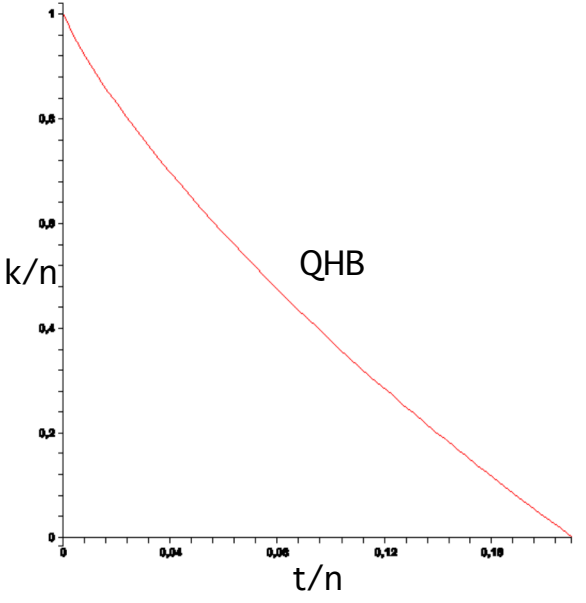
where E is now any operator acting on $2t$ qubits (that is, it replaces $E_a^\dagger E_b$ in (30)). This can be easily interpreted as saying that no measurement on $2t$ qubits can learn information about the codeword. Alternatively, it says we can *detect* up to $2t$ errors on the code without necessarily being able to say what those errors are. That is, we can distinguish those errors from the identity.

If the matrix C_{ab} in (30) has maximum rank, the code is called *nondegenerate*. If not, as for the nine-qubit code, the code is *degenerate*. In a degenerate code, different errors look the same when acting on the coding subspace.

For a nondegenerate code, we can set a simple bound on the parameters of the code simply by counting states. Each error E acting on each basis codeword $|\psi_i\rangle$ produces a linearly independent state. All of these states must fit in the full Hilbert space of n qubits, which has dimension 2^n. If the code encodes k qubits, and corrects errors on up to t qubits, then

$$\left(\sum_{j=0}^{t} 3^j \binom{n}{j}\right) 2^k \leq 2^n. \tag{35}$$

The quantity in parentheses is the number of errors of *weight* t or less: that is, the number of tensor products of I, X, Y, and Z that are the identity in all but t or fewer places. This inequality is called the *quantum Hamming bound*. While the quantum Hamming bound only applies to nondegenerate codes, we do not know of any codes that beat it.

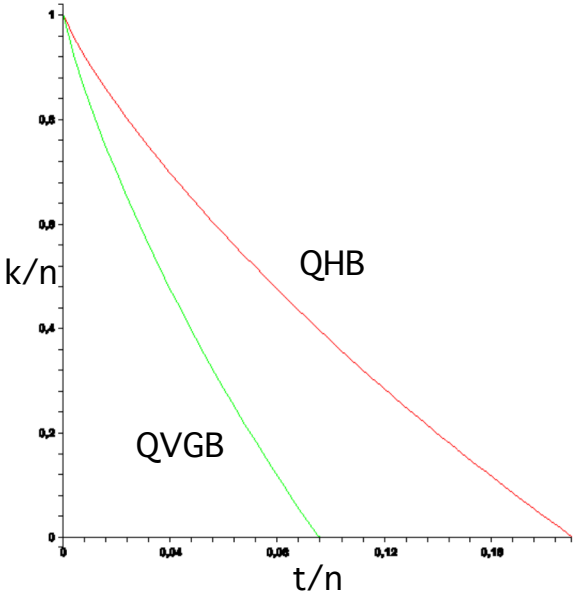

1
0,8
0,6
0,4
0,2
0
k/n
QHB
0
0,04
0,08
0,12
0,16
t/n

For $t = 1$, $k = 1$, the quantum Hamming bound tells us $n \geq 5$. In fact, there is a code with $n = 5$, which you will see later. A code that corrects t errors is said to have *distance* $2t + 1$, because it takes $2t + 1$ single-qubit changes to get from one codeword to another. We can also define distance as the minimum weight of an operator E that violates equation (34) (a definition which also allows codes of even distance). A quantum code using n qubits to encode k qubits with distance d is written as an $[[n, k, d]]$ code (the double brackets distinguish it from a classical code). Thus, the nine-qubit code is a $[[9, 1, 3]]$ code, and the five-qubit code is a $[[5, 1, 3]]$ code.

We can also set a lower bound telling us when codes exist. I will not prove this here, but an $[[n, k, d]]$ code exists when

$$\left(\sum_{j=0}^{d-1} 3^j \binom{n}{j}\right) 2^k \leq 2^n \tag{36}$$

(known as the quantum Gilbert-Varshamov bound [**3**]). This differs from the quantum Hamming bound in that the sum goes up to $d-1$ (which is equal to $2t$) rather than stopping at t.


1
0,8
0,6
0,4
0,2
0
k/n
QHB
QVGB
0
0,04
0,08
0,12
0,16
t/n

THEOREM 4. *A quantum $[[n, k, d]]$ code exists when (36) holds. Any nondegenerate $[[n, k, d]]$ code must satisfy (35). For large n, $R = k/n$ and $p = d/2n$ fixed, the best nondegenerate quantum codes satisfy*

$$1 - 2p \log_2 3 - H(2p) \leq R \leq 1 - p \log_2 3 - H(p), \tag{37}$$

where $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$.

One further bound, known as the Knill-Laflamme bound [**11**] or the quantum Singleton bound, applies even to degenerate quantum codes. For an $[[n, k, d]]$ quantum code,

$$n - k \geq 2d - 2. \tag{38}$$

This shows that the $[[5, 1, 3]]$ code really is optimal — a $[[4, 1, 3]]$ code would violate this bound.

I will not prove the general case of this bound, but the case of $k = 1$ can be easily understood as a consequence of the No-Cloning theorem. Suppose r qubits of the code are missing. We can substitute $|0\rangle$ states for the missing qubits, but there are r errors on the resulting codeword. The errors are of unknown type, but all the possibilities are on the same set of r qubits. Thus, all products $E_a^\dagger E_b$ in condition (30) have weight r or less, so this sort of error (an "erasure" error [**9**]) can be corrected by a code of distance $r+1$. Now suppose we had an $[[n, 1, d]]$ code with $n \leq 2d - 2$. Then we could split the qubits in the code into two groups of size at most $d - 1$. Each group would have been subject to at most $d - 1$ erasure errors, and could therefore be corrected without access to the other group. This would produce two copies of the encoded state, which we know is impossible.

4. Stabilizer codes

Now let us return to the nine-qubit code, and examine precisely what we need to do to correct errors.

First, we must determine if the first three qubits are all the same, and if not, which is different. We can do this by measuring the parity of the first two qubits and the parity of the second and third qubits. That is, we measure

$$Z \otimes Z \otimes I \text{ and } I \otimes Z \otimes Z. \tag{39}$$

The first tells us if an X error has occurred on qubits one or two, and the second tells us if an X error has occurred on qubits two or three. Note that the error detected in both cases anticommutes with the error measured. Combining the two pieces of information tells us precisely where the error is.

We do the same thing for the other two sets of three. That gives us four more operators to measure. Note that measuring $Z \otimes Z$ gives us just the information we want and no more. This is crucial so that we do not collapse the superpositions used in the code. We can do this by bringing in an ancilla qubit. We start it in the state $|0\rangle + |1\rangle$ and perform controlled-Z operations to the first and second qubits of the code:

$$(|0\rangle + |1\rangle) \sum_{abc} c_{abc}|abc\rangle \to \sum_{abc} c_{abc} \left(|0\rangle|abc\rangle + (-1)^{a\oplus b}|1\rangle|abc\rangle\right) \tag{40}$$

$$= \sum_{abc} c_{abc} \left(|0\rangle + (-1)^{\mathrm{parity}(a,b)}|1\rangle\right)|abc\rangle. \tag{41}$$

At this point, measuring the ancilla in the basis $|0\rangle \pm |1\rangle$ will tell us the eigenvalue of $Z \otimes Z \otimes I$, but nothing else about the data.

Second, we must check if the three signs are the same or different. We do this by measuring

$$X \otimes X \otimes X \otimes X \otimes X \otimes X \otimes I \otimes I \otimes I \tag{42}$$

and

$$I \otimes I \otimes I \otimes X \otimes X \otimes X \otimes X \otimes X \otimes X. \tag{43}$$

This gives us a total of 8 operators to measure. These two measurements detect Z errors on the first six and last six qubits, correspondingly. Again note that the error detected anticommutes with the operator measured.

This is no coincidence: in each case, we are measuring an operator M which should have eigenvalue $+1$ for any codeword:

$$M|\psi\rangle = |\psi\rangle. \tag{44}$$

If an error E which anticommutes with M has occurred, then the true state is $E|\psi\rangle$, and

$$M\left(E|\psi\rangle\right) = -EM|\psi\rangle = -E|\psi\rangle. \tag{45}$$

That is, the new state has eigenvalue -1 instead of $+1$. We use this fact to correct errors: each single-qubit error E anticommutes with a particular set of operators $\{M\}$; which set, exactly, tells us what E is.

In the case of the nine-qubit code, we cannot tell exactly what E is, but it does not matter. For instance, we cannot distinguish Z_1 and Z_2 because

$$Z_1 Z_2|\psi\rangle = |\psi\rangle \iff Z_1|\psi\rangle = Z_2|\psi\rangle. \tag{46}$$

This is an example of the fact that the nine-qubit code is degenerate.

Z	Z	I	I	I	I	I	I	I
I	Z	Z	I	I	I	I	I	I
I	I	I	Z	Z	I	I	I	I
I	I	I	I	Z	Z	I	I	I
I	I	I	I	I	I	Z	Z	I
I	I	I	I	I	I	I	Z	Z
X	X	X	X	X	X	I	I	I
I	I	I	X	X	X	X	X	X

TABLE 2. The stabilizer for the nine-qubit code. Each column represents a different qubit.

Table 2 summarizes the operators we measured. These 8 operators generate an Abelian group called the *stabilizer* of the nine-qubit code. The stabilizer contains all operators M in the Pauli group for which $M|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$ in the code.

Conversely, given an Abelian subgroup S of the Pauli group $\mathcal{P}_n$ (which, if you recall, consists of tensor products of I, X, Y, and Z with an overall phase of $\pm 1, \pm i$), we can define a quantum code $T(S)$ as the set of states $|\psi\rangle$ for which $M|\psi\rangle = |\psi\rangle$ for all $M \in S$. S must be Abelian and cannot contain -1, or the code is trivial: If $M, N \in S$,

$$MN|\psi\rangle = M|\psi\rangle = |\psi\rangle \tag{47}$$

$$NM|\psi\rangle = N|\psi\rangle = |\psi\rangle \tag{48}$$

so

$$[M, N]|\psi\rangle = MN|\psi\rangle - NM|\psi\rangle = 0. \tag{49}$$

Since elements of the Pauli group either commute or anticommute, $[M, N] = 0$. Clearly, if $M = -1 \in S$, there is no nontrivial $|\psi\rangle$ for which $M|\psi\rangle = |\psi\rangle$.

If these conditions are satisfied, there will be a nontrivial subspace consisting of states fixed by all elements of the stabilizer. We can tell how many errors the code corrects by looking at operators that commute with the stabilizer. We can correct errors E and F if either $E^\dagger F \in S$ (so E and F act the same on codewords), or if $\exists M \in S$ s.t. $\{M, E^\dagger F\} = 0$, in which case measuring the operator M distinguishes between E and F. If the first condition is ever true, the stabilizer code is degenerate; otherwise it is nondegenerate.

We can codify this by looking at the normalizer $N(S)$ of S in the Pauli group (which is in this case equal to the centralizer, composed of Pauli operators which commute with S). The distance d of the code is the minimum weight of any operator in $N(S) \setminus S$ [**3, 7**].

THEOREM 5. *Let S be an Abelian subgroup of order 2^a of the n-qubit Pauli group, and suppose $-1 \notin S$. Let d be the minimum weight of an operator in $N(S) \setminus S$. Then the space of states $T(S)$ stabilized by all elements of S is an $[[n, n-a, d]]$ quantum code.*

To correct errors of weight $(d-1)/2$ or below, we simply measure the generators of S. This will give us a list of eigenvalues, the *error syndrome*, which tells us whether the error E commutes or anticommutes with each of the generators. The error syndromes of E and F are equal iff the error syndrome of $E^\dagger F$ is trivial. For a nondegenerate code, the error syndrome uniquely determines the error E (up to a trivial overall phase) — the generator that anticommutes with $E^\dagger F$ distinguishes E from F. For a degenerate code, the error syndrome is not unique, but error syndromes are only repeated when $E^\dagger F \in S$, implying E and F act the same way on the codewords.

If the stabilizer has a generators, then the code encodes $n - a$ qubits. Each generator divides the allowed Hilbert space into $+1$ and -1 eigenspaces of equal sizes. To prove the statement, note that we can find an element G of the Pauli group that has any given error syndrome (though G may have weight greater than $(d - 1)/2$, or even greater than d). Each G maps $T(S)$ into an orthogonal but isomorphic subspace, and there are 2^a possible error syndromes, so $T(S)$ has dimension at most $2^n/2^a$. In addition, the Pauli group spans $U(2^n)$, so its orbit acting on any single state contains a basis for $\mathcal{H}$. Every Pauli operator has *some* error syndrome, so $T(S)$ has dimension exactly 2^{n-a}.

5. Some other important codes

Stabilizers make it easy to describe new codes. For instance, we can start from classical coding theory, which describes a linear code by a generator matrix or its dual, the parity check matrix. Each row of the generator matrix is a codeword, and the other codewords are all linear combinations of the rows of the generator matrix. The rows of the parity check matrix specify parity checks all the classical codewords must satisfy. (In quantum codes, the stabilizer is closely analogous to the classical parity check matrix.) One well-known code is the seven-bit Hamming code correcting one error, with parity check matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{50}$$

If we replace each 1 in this matrix by the operator Z, and 0 by I, we are really changing nothing, just specifying three operators that implement the parity check measurements. The statement that the classical Hamming code corrects one error is the statement that each bit flip error of weight one or two anticommutes with one of these three operators.

Z	Z	Z	Z	I	I	I
Z	Z	I	I	Z	Z	I
Z	I	Z	I	Z	I	Z
X	X	X	X	I	I	I
X	X	I	I	X	X	I
X	I	X	I	X	I	X

TABLE 3. Stabilizer for the seven-qubit code.

Now suppose we replace each 1 by X instead of Z. We again get three operators, and they will anticommute with any weight one or two Z error. Thus, if we make a stabilizer out of the three Z operators and the three X operators, as in table 3, we get a code that can correct any single qubit error [**18**]. X errors are picked up by the first three generators, Z errors by the last three, and Y errors are distinguished by showing up in both halves. Of course, there is one thing to check: the stabilizer must be Abelian; but that is easily verified. The stabilizer has 6 generators on 7 qubits, so it encodes 1 qubit — it is a $[[7, 1, 3]]$ code.

In this example, we used the same classical code for both the X and Z generators, but there was no reason we had to do so. We could have used any two classical codes C_1 and C_2 [**5, 19**]. The only requirement is that the X and Z generators commute. This corresponds to the statement that $C_2^\perp \subseteq C_1$ ($C_2^\perp$ is the dual code to C_2, consisting of those words which are orthogonal to the codewords of C_2). If C_1 is an $[n, k_1, d_1]$ code, and C_2 is an $[n, k_2, d_2]$ code (recall single brackets means a classical code), then the corresponding quantum code is an $[[n, k_1+k_2-n, \min(d_1, d_2)]]$ code.[2] This construction is known as the CSS construction after its inventors Calderbank, Shor, and Steane.

The codewords of a CSS code have a particularly nice form. They all must satisfy the same parity checks as the classical code C_1, so all codewords will be superpositions of words of C_1. The parity check matrix of C_2 is the generator matrix of $C_2^\perp$, so the X generators of the stabilizer add a word of $C_2^\perp$ to the state. Thus, the codewords of a CSS code are of the form

$$\sum_{w \in C_2^\perp} |u + w\rangle, \tag{51}$$

where $u \in C_1$ ($C_2^\perp \subseteq C_1$, so $u + w \in C_1$). If we perform a Hadamard transform

$$|0\rangle \longleftrightarrow |0\rangle + |1\rangle \tag{52}$$

$$|1\rangle \longleftrightarrow |0\rangle - |1\rangle \tag{53}$$

on each qubit of the code, we switch the Z basis with the X basis, and C_1 with C_2, so the codewords are now

$$\sum_{w \in C_1^\perp} |u + w\rangle \quad (u \in C_2). \tag{54}$$

Thus, to correct errors for a CSS code, we can measure the parities of C_1 in the Z basis, and the parities of C_2 in the X basis.

Another even smaller quantum code is the $[[5, 1, 3]]$ code I promised earlier [**2, 13**]. Its stabilizer is given in table 4. I leave it to you to verify that it commutes

X	Z	Z	X	I
I	X	Z	Z	X
X	I	X	Z	Z
Z	X	I	X	Z

TABLE 4. The stabilizer for the five-qubit code.

and actually does have distance 3.

A representation of stabilizers that is often useful is as a pair of binary matrices, frequently written adjacent with a line between them [**3**]. The first matrix has a 1 everywhere the stabilizer has an X or a Y, and a 0 elsewhere; the second matrix has a 1 where the stabilizer has a Y or a Z. Multiplying together Pauli operators corresponds to adding the two rows for both matrices. Two operators M and N commute iff their binary vector representations $(a_1|b_1)$, (a_2, b_2) are orthogonal under a symplectic inner product: $a_1 b_2 + b_1 a_2 = 0$. For instance, the stabilizer for the five-qubit code becomes the matrix

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right). \tag{56}$$