A Crash Course on Coding Theory

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Disclaimer

This is an opinionated survey of coding theory, unbiased by actual reading of papers.

Trivial Constructions

(Think binary, then generalize)

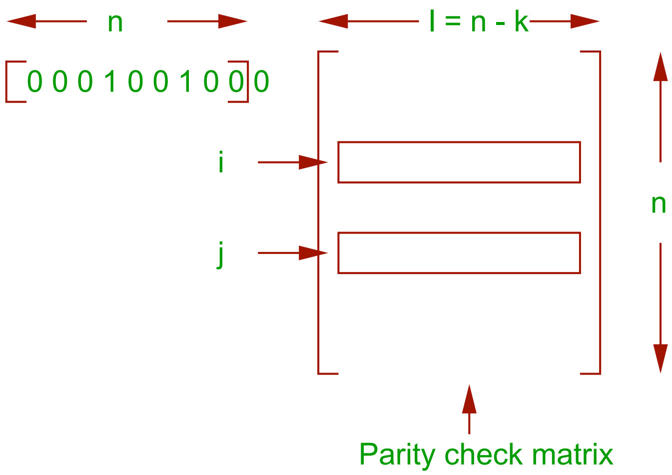
- Trivial code:
  - E is the identity function.
  - Has $n = k, d = 1$.
  - Generalizes to all alphabets!
- Parity code:
  - Append parity of all k bits to message.
  - Gives $n = k + 1, d = 2$.
  - More generally, append sum of the first k letters.

Meet Singleton bound: $k + d \leq n + 1$.

Hamming code

- Historically first (approximately).
- For any l, $[n = (q^l - 1)/(q - 1), n - l, d = 3]_q$ code.
- Rows of parity check matrix H:
  - All non-zero vectors of length l,
  - Scalar multiples removed (say by fixing first non-zero entry to 1).
- Since any two rows of H are linearly independent, distance is greater than 2.

Hamming code (contd).



Hadamard code

$[n = q^l, k = l, d = q^l - q^{l-1}]_q$ code.

(Roughly, the dual of the Hamming code.)

Construction:

- Message: $m = \langle m_1, \ldots, m_k \rangle$
  associated with
  $$M(x_1, \ldots, x_k) = \sum_{i=1}^k m_i x_i.$$
- Encoding: $E(m) = \langle M(x) \rangle_{x \in \Sigma^k}$.
- Distance = Why?

Polynomials over finite fields

Some facts (Fix size of field to q).

- Non-zero deg. $\leq l$ poly. has $\leq l$ zeroes. (alt'ly, zero on $\leq l/q$ fraction of inputs.)
- Deg $\leq l$ polys $=$ vector space of dim. $l+1$.
- Non-zero deg. $\leq l$, m-variate, poly. zero on $\leq l/q$ fraction of inputs.
- $l < q \Rightarrow$ Deg. $\leq l$, m-variate, polys $=$ dim. $\binom{m+l}{l}$ vector space.

Poly facts (contd.)

- Non-zero deg. $\leq l$, m-var., poly. zero on $\leq 1 - q^{-(l/(q-1))}$ fraction of inputs.
- Vector space of dimension $\geq \binom{m}{l}$.
- Actual dimension $=$ # of ordered partitions of l into integers from $\{0, \ldots, q-1\}$.

Hadamard codes (contd).

- Codewords are evaluations of degree 1 polynomials over $\mathbb{F}_q$.
- May agree in at most $1/q$ fraction of indices.
- $\Rightarrow$ Distance $\geq q^l - q^{l-1}$.

Reed-Solomon Codes

Reed-Solomon Codes:

$[n, k, n-k+1]_q$ code for $q \geq n$.

- Fix distinct $x_0, \ldots, x_{n-1} \in \Sigma$.
- Message: Coefficients of polynomials $\langle m_0, \ldots, m_{k-1} \rangle \approx M(x) = \sum_{i=0}^{k-1} m_i x^i$
- Encoding: Evaluations of polynomials $\langle M(x_0), \ldots, M(x_{n-1}) \rangle$
- Distance follows from fact on univariate polynomials.

Reed-Muller Codes

Codes based on multiv. polynomials.

variables $= m$; degree $\leq r$.

Coding theory favorite: $q = 2$, $[n, k, d]_2$ code

$$n = q^m; k = \left(\binom{m}{r}\right); d = q^{m-r}$$

Complexity th. favorite: $q > r$, $[n, k, d]_q$ code

$$n = q^m; k = \binom{m+r}{r}; d = q^m - rq^{m-1}$$

Latter version:

Larger alphabet; larger distance.

Can also take indiv. degree bounded polys.

Random linear codes

Pick $c_1, \ldots, c_k \in_R \Sigma^n$ and let

$$G = \begin{bmatrix} - & - & c_1 & - & - \\ - & - & c_2 & - & - \\ & & \vdots & & \\ - & - & c_k & - & - \end{bmatrix}$$

Analysis (of Distance):

- For fixed $\langle \alpha_1, \ldots, \alpha_k \rangle \neq \vec{0}$
  $\Pr\left[\alpha G \in B(\vec{0}, d)\right] \leq q^{(H_q(d/n)-1)n}$.
- Thus
  $\Pr\left[\exists \alpha \text{ s.t.} \alpha G \in B(\vec{0}, d)\right] \leq q^{k+(H_q(d/n)-1)n}$.
- Thus if $k/n < 1 - H_q(d/n)$
  then code is $[n, k, d]_q$ code.

Hamming Balls

- Recall $B(\vec{x}, r)$ ball of radius r around $\vec{x}$.
- $V(n, r, q) =$ "volume" of $B(\cdot, r)$ in Σ^n.
- Let $H_q(p)$ be q-ary entropy function.

$$H_q(p) = p \log_q \left(\frac{q-1}{p}\right) + (1-p) \log_q \left(\frac{1}{1-p}\right)$$

Fact:

$$V(n, pn, q) \approx q^{H_q(p)n}$$

Summary

- Reed-Solomon codes are great, but alphabet is too large.
- Hadamard codes are exponentially large but have great distance.
- Random codes are great.
  Achieve $k/n, d/n > 0$ over binary alphabet.

  But non-constructive; non-verifiable;
  non-decodable.

Operations on codes

Can produce codes from other codes by some basic operations.

- Puncturing:
  Throw away column of generator matrix.
  $[n,k,d]_q \quad \to \quad [n-1,k,d-1]_q$
  Asymptotically weaker.
  (Every linear code is punctured Hadamard code.)

- Pasting:
  Adjoin generators of codes of same dim.
  to get longer code.
  $[n_1,k,d_1]_q \mid [n_2,k,d_2]_q$
  $\to \quad [n_1+n_2,k,d_1+d_2]_q$
  Asymptotically weaker.

Direct Products

- $[n_1,k_1,d_1]_q \otimes [n_2,k_2,d_2]_q$
  $\to \quad [n_1n_2,k_1k_2,d_1d_2]_q$

- Let R generate $[n_1,k_1,d_1]$ code.
  Let C generate $[n_2,k_2,d_2]$ code.

- Codewords of $R \otimes C$ are $n_1 \times n_2$ matrices:
  $\{C^T X R \mid X \in \Sigma^{k_1 \times k_2}\}$

- Columns of tensor are codewords of C.
  Rows of tensor are codewords of R.

- Asymptotically weakening.

Example: tensor product of RS codes, gives bivariate polynomials of degree k_1-1 in x and k_2-1 in y.

Concatenation of codes [Forney]

$$[n_1, k_1, d_1]_{q^{k_2}} \circ [n_2, k_2, d_2]_q \\ \to \quad [n_1 n_2, k_1 k_2, d_1 d_2]_q.$$

- Compare with Tensor Products!
- Terminology: First code is "outer code"
  Second code is "inner code".
- Encoding:
  Encode message with outer encoder.
  Then encode each letter w. inner code.
- Linearity achieved with care. Outer alphabet must be properly extended from inner alphabet.

Example: RS $\circ$ Hadamard

- Fix k_2.
- Let $n_1 = 2^{k_2}, k_1 = \cdot 5 n_1, q = 2$.
- RS outer code: $[n_1, \cdot 5 n_1, \cdot 5 n_1]_{n_1}$
- Hadamard inner code: $[n_1, k_2, \cdot 5 n_1]_2$
- Concatenate code: $[n_1^2, \cdot 5 k_2 n_1, \cdot 25 n_1^2]_2$
- Let $n = n_1^2$, Use $k_2 = \cdot 5 \log_2 n$
  $[n, \cdot 25 \sqrt{n} \log_2 n, \cdot 25 n]_2$ code
- Constant distance, poly rate!
  Good for many complexity th. applications.

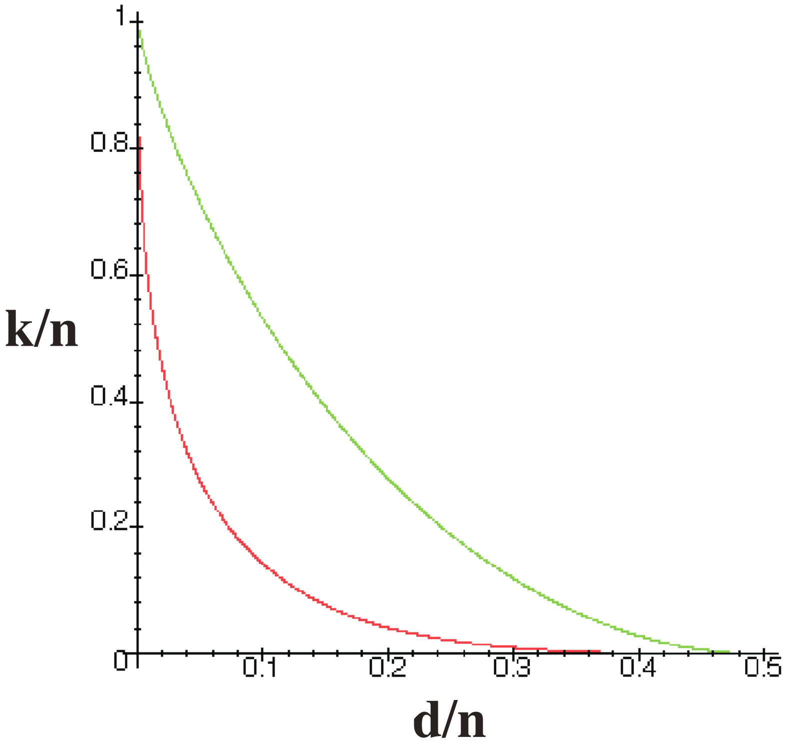
Forney Codes

- Concat. RS codes with random linear code.
- At each level code has constant $d/n, k/n$.
- Concat. code has constants for both ratios.

Thm: (Unflattering version). Asymptotically good code can be found in quasi-polynomial time.

Thm: (Flattering version). By using 2 levels of concatenation, asymptotically good code can be found in nearly linear time, with polylog space.

(Not the end of story.)



Justesen Codes

(More "explicit" codes; Nice idea; Exposition due to Zuckerman)

Suppose: Can explicitly describe sample space containing n_1 codes such that all but ϵ fraction of the codes are $[n_2, k_2, d_2]_q$ codes.

Then concatenate codes as follows:

- Encode message using $[n_1, k_1, d_1]_{q^{k_2}}$ code.
- Encode ith letter of result using ith code from sample space.
- Result is a $[n_1 n_2, k_1 k_2, (d_1 - \epsilon n_1) d_2]_q$ code.

Can get asymptotically good code!

Justesen's sample space

- The Wozencraft ensemble.
- Let $n_1 = q^{k_2}$.
- Let $F = GF(q^{k_2})$.
- Message for inner code: $x \in F$.
- α-th code maps $x \mapsto \langle x, \alpha x \rangle$.
- For most α, get $[2k_2, k_2, H_q^{-1}(\frac{1}{2})(2k_2)]_q$ code.
- (I.e., most codes, as good as random code!)

Wozencraft ensemble (contd).

(Ignoring subscript on k_2 below.)

α is d-bad if α-th code not $[2k, k, d]_q$ code.

Claim: # of bad α's is at most

$$V(2k, d, q) \approx q^{H_q(d/(2k))\cdot(2k)}.$$

Proof:

- If $\alpha_1 \neq \alpha_2$ then intersection of corr. codes is the 0-vector.
- Each bad code must have non-zero vector in $B(\vec{0}, d)$. These must be distinct.
- Thus, at most $V(2k, d, q)$ bad codes.

Further pointers

- Weldon codes: $x \mapsto \langle x, \alpha x, \alpha^2 x, \ldots \rangle$.
- Gets distance arbitrarily close to $1 - \frac{1}{q}$.
- Alternate route: Can apply Zuckerman exposition with 2-level concatenation and random linear codes.
- Sugiyama et al. papers: Get better rates than Weldon.