Shafi Goldwasser, Michael Sipser

Private Coins versus Public Coins in Interactive Proof Systems

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Definition: An *Interactive proof protocol* is given by two functions:

$$V: \Sigma^* \times \Sigma^* \times \Sigma^* \rightarrow \Sigma^* \bigcup \{\textbf{accept}, \textbf{reject}\}$$

$$P: \Sigma^* \rightarrow \Sigma^*$$

Let s_i denote the concatenation of i pairs of messages, $s_i = \#x_1\#y_1\# \cdots \#x_i\#y_i$. We write $V(w,r,s_i) = x_{i+1}$ to mean that V on input w, with random sequence r, and current message stream ~~s~~ s_i produces next message x_{i+1}. We say $P(s_i\#x_{i+1}) = y_{i+1}$ to mean that P produces next message y_{i+1} given current message stream $s_i\#x_{i+1}$. The exchange of a single pair of messages is called a *round*.

(x_1 can always contain w)

For a given input w and random sequence r we say

$$(V^*P)(w,r) \text{ accepts}$$

if there exists a message stream $s = \#x_1\#y_1\# \cdots \#x_l\#y_l$ such that $V(w,r,s) = \textbf{accept}$, and for each $i < l$, $V(w,r,s_i) = x_{i+1}$ and $P(s_i\#x_{i+1}) = y_{i+1}$.

Let us assume for simplicity that there is a function l such that for inputs w of length n, V will only accept if the length of r is $l(n)$. Then we write

$$\Pr[(V^*P)(w) \text{ accepts}]$$

to mean $\Pr[(V^*P)(w,r) \text{ accepts}]$ for r chosen (uniformly) randomly from $\Sigma^{\cancel{l(|x|)}}$. $\Sigma^{l(|w|)}$

Further we let

$$\Pr[V(w) \text{ accepts}]$$

denote $\max_P \Pr[(V^*P)(w) \text{ accepts}]$.

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Let the language of the verifier, $L(V)=$

$$\{w\colon \Pr[V(w) \text{ accepts}] > 1/2\}$$

Say V has error probability e if for all $w \in \Sigma^*$:

1) if $w \in L(V)$, $\Pr[V(w) \text{ accepts}] \geq 1-e$
2) if $w \notin L(V)$, $\Pr[V(w) \text{ accepts}] \leq e$

For $W \subseteq \Sigma^*$, we say $W \in \mathrm{IP}$ if there is a polynomial time verifier V with error probability 1/3 accepting W. As we shall see later, the class IP is unaffected if we substitute e for 1/3, where ~~$2^{-\mathrm{poly}(n)} \leq e \leq 1/2 - 2^{-\mathrm{poly}(n)}$~~.

$2^{-\mathrm{poly}(n)} \leq e \leq 1/2 - 1/\mathrm{poly}(n)$

Definition: An *Interactive proof protocol with public coin* is defined as above with the following difference. The random input r is considered to be the concatenation of l strings $r = r_1 r_2 \cdots r_l$ where l is the number of rounds and V is restricted to produce r_i as it's i^{th} message, i.e., for $i \leq l$, $V(w, r, s_i) = r_i$ or **accept** or **reject**.

This notion is essentially identical to that of the Arthur-Merlin game defined by Babai in [B]. Following his terminology we say that for $W \subseteq \Sigma^*$, $W \in$ AM(poly) if $W \in$ IP as above and the interactive proof protocol uses a public coin. We refer to an Arthur-Merlin game as an A-M protocol.

For polynomial Q, say $W \in IP[Q(n)]$ if $W \in IP$ with a verifier which never sends more than $Q(n)$ messages for inputs of length n. Similarly define $AM[Q(n)]$.

4.1. Approximate lower bound lemma

This lemma, an application of Carter-Wegman universal hashing [CW], due to Sipser [Si], plays a key role in our proof of equivalence. Its application to approximate lower bounds was first given by Stockmeyer [St]. Its application in Arthur-Merlin protocols first appears in Babai [B].

Definition: Let D be a $k \times b$ Boolean matrix. The *linear function* $h_D: \Sigma^k \rightarrow \Sigma^b$ is given by $h_D(x) = x \cdot D$ using ordinary matrix multiplication modulo 2. A *random linear function* is obtained by selecting the matrix D at random. If $H = \{h_1, \ldots, h_l\}$ is a collection of functions, $C \subseteq \Sigma^k$, and $\cancel{D} \subseteq \Sigma^b$ then $H(C)$ denotes $\bigcup h_i(C)$, and $H^{-1}(\cancel{D})$ denotes $\bigcup h_i^{-1}(\cancel{D})$. Let $|C|$ denote the cardinality of C.

Z
Z
Z

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Lemma: Given $b,k,l>0$, $l>\max(b,8)$, and $C\subseteq\Sigma^k$. Randomly select l linear functions $H=\{h_1,\ldots,h_l\}$, $h_i:\Sigma^k\rightarrow\Sigma^b$ and l^2 strings $Z=\{z_1,\ldots,z_{l^2}\}\subseteq\Sigma^b$. Then

1. If $b=2+\lceil\log|C|\rceil$ then
   a) $\Pr[|H(C)| \geq |C|/l] \geq 1-2^{-l}$
   b) $\Pr[H(C)\cap Z \neq \emptyset] \geq 1-2^{-l/8}$
2.
   a) $|H(C)| \leq l|C|$
   b) If for $d>0$, $|C| \leq 2^b/d$ then:
   $\Pr[H(C)\cap Z\neq\emptyset] \leq l^3/d$

if $2^b/4 \geq |C| \geq 2^b/8$ **then** $\Pr[H(C)\cap Z=\emptyset] \leq 2^{-l/8}$
if $|C| \leq 2^b/d$, $d>0$, **then** $\Pr[H(C)\cap Z\neq\emptyset] \leq l^3/d$

1. If $b = 2 + \lceil \log |C| \rceil$ then
   a) $\Pr[|H(C)| \geq |C|/l] \geq 1 - 2^{-l}$
   b) $\Pr[H(C) \cap Z \neq \emptyset] \geq 1 - 2^{-l/8}$

Proof 1a: Since $2^b \geq 4|C|$ the following chain of statements are easily verified. Let $(h_i(x))^j$ denote the jth bit of the string $h_i(x)$. Fix $x, y \in \Sigma^k$, $x \neq y$, $i, j > 0$, except where quantified.

$\Pr[(h_i(x))^j = (h_i(y))^j] = 1/2$

$\Pr[h_i(x) = h_i(y)] = 2^{-b}$

$\Pr[\exists y \in C \ (x \neq y \,\&\, h_i(x) = h_i(y))] \leq |C| \cdot 2^{-b} \leq 1/4$

$\Pr[\forall i \leq l \ \exists y \in C \ (x \neq y \,\&\, h_i(x) = h_i(y))] \leq 4^{-l}$

$\Pr[\exists x \in C \ \forall i \leq l \ \exists y \in C \ (x \neq y \,\&\, h_i(x) = h_i(y))]$
$\leq |C| \cdot 4^{-l} \leq 2^{-l}$

Therefore $\Pr[|H(C)| \geq |C|/l] \geq 1 - 2^{-l}$

1. If $b = 2 + \lceil \log |C| \rceil$ then
   a) $\Pr[|H(C)| \geq |C|/l] \geq 1 - 2^{-l}$
   b) $\Pr[H(C) \cap Z \neq \emptyset] \geq 1 - 2^{-l/8}$

Proof 1b: Since $|C| \geq 2^b/8$, if $|H(C)| \geq |C|/l$ then

$$\frac{|H(C)|}{|\Sigma^b|} \geq \frac{1}{8l}$$

Thus it is likely that one of the l^2 strings in Z will be in $H(C)$.

$$\Pr[H(C) \cap Z = \emptyset] \leq (1 - 1/8l)^{l^2} + 2^{-l} < 2^{-l/8}$$

2.

a) $|H(C)| \leq l|C|$

b) If for $d>0$, $|C| \leq 2^b/d$ then:

$\Pr[H(C) \cap Z \neq \emptyset] \leq l^3/d$

Proof 2a: Obvious.

Proof 2b: Since

$$\frac{|H(C)|}{\Sigma^b} \leq \frac{l|C|}{d|C|} = \frac{l}{d}$$

The probability that each z_i is in $H(C)$ is at most l/d. Thus the probability that any of the l^2 strings in Z is in $H(C)$ is at most l^3/d.

■

We use this lemma to obtain Arthur-Merlin protocols for showing an approximate lower bound on the size of sets. Let C be a set in which Arthur can verify membership, possibly with Merlin's help. Then let Arthur picks random H and Z and Merlin attempt to respond with $x \in C$ such that some $x \in H^{-1}(z)$. If C is large then he will likely succeed and if C is small he will likely fail.

if $2^b/4 \geq |C| \geq 2^b/8$ **then** $\Pr[H(C) \cap Z = \emptyset] \leq 2^{-l/8}$

if $|C| \leq 2^b/d$, $d>0$, **then** $\Pr[H(C) \cap Z \neq \emptyset] \leq l^3/d$

4.2. Main Theorem

Theorem: $\mathrm{IP}[Q(n)] = \mathrm{AM}[Q(n)+2]$ for any polynomial $Q(n)$

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An informal proof sketch: Let's focus on 1-round protocols. Assume V has an exponentially small error probability e, sends only messages of length m, and uses random sequences of length l. For each $x \in \Sigma^m$ let $\beta_x = \{r\colon V(\cancel{r,w,}\#) = x\}$. For every $y \in \Sigma^m$ let $\alpha_{xy} = \{r\colon r \in \beta_x \quad \& \quad V(\cancel{r,w,}\#x\#y) = \textbf{accept}\}$. Clearly, for each x, the optimal prover will select a y_x maximizing $|\alpha_{xy}|$. Let $\alpha_x = \alpha_{xy_x}$. Let $\alpha_0 = \bigcup_x \alpha_x$. Then $\Pr[V(w) \text{ accepts}] = |\alpha_0|/2^l$.

$V(w,r,\#)$

$V(w,r,\#x\#y)$

NOTA: the α_x are disjoint.

We next present the protocol by which A and M simulate V and P. M tries to convince A that $|\alpha_0| > e \cdot 2^l$ because this implies that $\Pr[V(w) \text{ accepts}] > e$ and hence ≈ 1. He does this by showing that there are "many" α_x's which are "large", where "many" $\times$ "large" $> e \cdot 2^l$. The tradeoff between "many" and "large" is governed by a parameter b sent by M to A.

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More precisely, M first sends b to A. Then two approximate lower bound protocols ensue. The first convinces A that

$e2^l/2^b$ $|\{x: |\alpha_x| \geq \cancel{2^b/(e \cdot 2^l)}\}| \geq 2^b$. M produces an x in that set as per the approximate lower bound lemma. The second convinces A that x really is in that set as claimed, i.e., that

$e2^l/2^b$ $|\alpha_x| \geq \cancel{2^b/(e \cdot 2^l)}$.

For g-round protocols iterate the first approximate lower bound protocol to obtain $\alpha_0 \supseteq \alpha_1 \supseteq \cdots \supseteq \alpha_g$ where there are "many$_i$" ways to extend α_{i-1} to α_i and α_g is "large". Require that $(\Pi \text{"many}_i\text{"}) \times \text{"large"} \geq e \cdot 2^l$.

Full proof: Let $W \in \mathrm{IP}[Q(n)]$. We may assume, without loss of generality, that on inputs w of length n there are exactly $g(n)=Q(n)/2$ pairs of messages sent between V and P, these messages are exactly $m(n)$ long and the random input r to V is $l(n)$ long. Let $e(n)$ bound the error probability.

Amplification Lemma: Let $p(n)$ be a polynomial. Let V be a verifier which on inputs of length n a total of at most $g(n)$ messages, each of length $m(n)$, using $l(n)$ random bits, and with error probability at most 1/3. Then there is a V' such that $L(V)=L(V')$, with a total of at most $g(n)$ messages, each of length $O(p(n)m(n))$, using $O(p(n)l(n))$ random bits and with an error probability of at most $2^{-p(n)}$.

proof: V' performs $O(p(n))$ independent parallel simulations of V and takes the majority vote of the outcomes. Details left to the reader. ■

(Full proof)

Let $X = \sum_i X_i$ be a sum of independent random indicator variables X_i. For each i, let $p_i = \Pr[X_i = 1]$, and let $\mu = \mathrm{E}[X] = \sum_i \mathrm{E}[X_i] = \sum_i p_i$.

Chernoff Bound (Upper Tail).

$$\Pr[X > (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu} \textit{ for any } \delta > 0.$$

$$\boxed{\Pr[X > (1+\delta)\mu] < e^{-\mu\delta^2/3}} \quad \text{for any } 0 < \delta < 1.$$

Chernoff Bound (Lower Tail).

$$\Pr[X < (1-\delta)\mu] < \left(\frac{e^{\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu} < e^{-\mu\delta^2/2} \textit{ for any } \delta > 0.$$

By this lemma we may assume:

$$e(n) \leq l(n)^{-12g^2(n)}$$

Further we may assume that $l(n) > \max(g(n), m(n), 80)$. We write g, m, e, l for $g(n)$, $m(n)$, $e(n)$ and $l(n)$ where n is understood.

(Full proof)

We now describe the functions A and M, simulating V and P, informally as two parties exchanging messages. The variables x_i and y_i represent messages sent by V and P respectively. In essence, the idea is for A to use the random hash functions to force M to produce a generic run of the V,P protocol and then finally to prove that this run would likely cause V to accept. The numbers b_i that M produces roughly correspond to the log of the number of possible generic messages that V can make at round i.

(Full proof)

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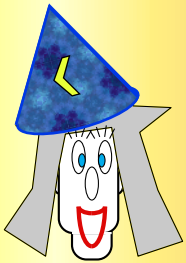
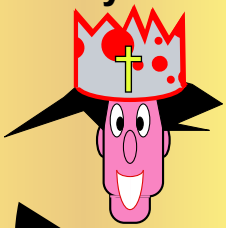
Arthur-Merlin Games

Poly-Time





$w \in L$

b_1 →

← H,Z

$1 \le i \le g$

x_i, y_i, b_{i+1} →

(via $s_{i-1}\#x_i$ many r would lead V to accept)

$x_i \in H^{-1}(Z)$?

← H,Z

r →

(many r would lead V to accept)

$r \in H^{-1}(Z)$?

$1 \le i \le g$, $V(w,r,s_{i-1}) = x_i$?

$V(w,r,s_g)$ = "accept" ?

$\sum b_i \ge l - g \log l$?

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Arthur-Merlin Games

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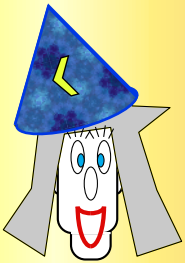

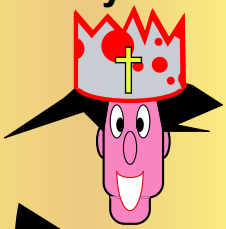

$w \in L$

NOTA: $[x]$=ceiling(x)

$\boldsymbol{b_1}=2+[\log|\gamma_{max}|]$

$\boldsymbol{H} \in_R (\Sigma^m \rightarrow \Sigma^{b_1})^l$, $\boldsymbol{Z} \in_R (\Sigma^m)^{l^2}$

$1 \le i \le g$

$\boldsymbol{x_i} \in H^{-1}(Z)$, $\alpha_{x_i} \in \gamma_{max}$, $\boldsymbol{y_i}=P(s_{i-1}\#x_i)$, $\boldsymbol{b_{i+1}}=2+[\log|\gamma_{max}|]$

$x_i \in H^{-1}(Z)$?

$\boldsymbol{H} \in_R (\Sigma^m \rightarrow \Sigma^{b_{i+1}})^l$, $\boldsymbol{Z} \in_R (\Sigma^m)^{l^2}$

$\boldsymbol{r}$

$r \in H^{-1}(Z)$?

$1 \le i \le g$, $V(w,r,s_{i-1}) = x_i$?

$V(w,r,s_g)$ = "accept" ?

$\sum b_i \ge l - g \log l$?

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Arthur's protocol

Round 0:

A initially makes a null move and receives number b_1 from M. Go to round 1.

Round i ($1 \le i \le g$):

So far A has received $b_1, \ldots, b_i$, and strings $x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1}$ from M. Now A randomly selects l linear functions $H = \{h_1, \ldots, h_l\}$, $h_i : \Sigma^m \to \Sigma^{b_{i+1}}$ and l^2 strings $Z = \{z_1, \ldots, z_{l^2}\} \subseteq \Sigma^{b_{i+1}}$ and them sends to M. A then expects to receive strings x_i and y_i and number b_{i+1} from M. A checks that $x_i \in H^{-1}(Z)$. If not then A immediately rejects. Then A performs round $i+1$.

$\sum^{b_i}$

$\sum^{b_i}$

Final round $g+1$:

Let $s_i = x_1 \# y_1 \# \cdots \# x_i \# y_i$. A randomly selects l linear functions $H = \{h_1, \ldots, h_l\}$, $h_i : \Sigma^l \to \Sigma^{b_{g+1}}$ and l^2 strings $Z \subseteq \Sigma^{b_{g+1}}$. It then expects to receive a string $r \in \Sigma^l$ from M and checks that $r \in H^{-1}(Z)$. A accepts if for each $i \le g$ $V(w, r, s_i) = x_{i+1}$, $V(w, r, s_g) =$ **accept** and $\sum b_i \ge l - g \log l$.

(and sends to M)

(Full proof)

Can Merlin convince Arthur?

Now we show that $\Pr[V(w)$ accepts$] > e(n)$ iff $\Pr[A(w)$ accepts$] \geq 2/3$.

($\rightarrow$) Merlin's protocol when $w \in W$

First some notation. For $r \in \Sigma^l$ and $s = v_1 \# v_2 \# \cdots \# v_k$ a stream of messages we say

$$(V^*P)(w,r) \text{ accepts via } s$$

if the first k messages sent by V and P agree with s and $(V^*P)(w,r)$ accepts.

Suppose $\Pr[V(w)$ accepts$] \geq 2/3$. Fix any P such that $\Pr[(V^*P)(w)$ accepts$] \geq 2/3$. We now exhibit a protocol for M such that $\Pr[(A^*M)(w)$ accepts$] \geq 2/3$.

(Full proof $w \in$ L)

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$1 \le i \le g$

Merlin's protocol

Obtain b_i ($i \le g$): Let $s_{i-1} = \#x_1\#y_1\# \cdots \#x_{i-1}\#y_{i-1}$ be the message stream for the V-P protocol produced so far. For each $x \in \Sigma^m$ let $\alpha_x = \{r: (V^*P)(w,r)$ accepts via $s_{i-1}\#x\}$. Group these α's into l classes $\gamma_1, \ldots, \gamma_l$ where γ_d contains α's of size $>2^{d-1}$ and $\le 2^d$. Choose the class $\gamma_{\max}$ whose union $\bigcup \gamma_{\max} = \bigcup \{\alpha_x: \alpha_x \in \gamma_{\max}\}$ is largest. Send $b_i = 2 + \lceil \log |\gamma_{\max}| \rceil$.

Round i:

M receives $h_1, \ldots, h_l$ from A and strings $z_1, \ldots, z_{l^2}$. If there is an $x \in H^{-1}(Z)$ such that $\alpha_x \in \gamma_{\max}$, call it x_i. Then, M responds with the pair x_i, y_i where $y_i = P(s_{i-1}\#x_i)$. Otherwise M responds with "failure". In the later analysis we refer to the set α_{x_i} as α_i. Set $i \leftarrow i+1$. Goto "obtain b_i".

(Full proof $w \in$ L)

Merlin's protocol

Obtain b_{g+1}: M produces the value b_{g+1} as follows: Let $s_g = s_{g-1}\#x_g\#y_g$ be the message stream that has been selected. So $\alpha_g = \{r: (V^*P)(w,r)$ accepts via $s_g\}$. Send $b_{g+1} = 2 + \lceil \log|\alpha_g| \rceil$.

Round $g+1$:

M receives $h_1, \ldots, h_l$ and strings $z_1, \ldots, z_{l^2} \in \Sigma^{b_{g+1}}$. If there is an $r \in \alpha_g \cap H^{-1}(Z)$, then M responds with r. Otherwise M responds with "failure". (Note that $r \in \alpha_g$ implies that $V(w,r,s_g) =$ **accept**)

End of Protocol.

(Full proof $w \in$ L)

We now show that $\Pr[(A^*M)(w)$ accepts$] \geq 2/3$. Let $\alpha_0 = \{r: (V^*P)(w,r) = \text{accept}\}$. Since $\Pr[V$ accepts $w]$ is high, $|\alpha_0| \geq (2/3)2^l$. By the definition of M, A will accept provided M never responds "failure" and $\sum b_i \geq l - g\log l$. By the approximate lower bound lemma the probability that M responds failure at any round is $\leq 2^{-l/8}$. Hence, the probability that M ever responds failure is $\leq g2^{-l/8} << 1/3$.

if $2^b/4 \geq |C| \geq 2^b/8$ **then** $\Pr[H(C) \cap Z = \emptyset] \leq 2^{-l/8}$

(Full proof $w \in L$)

The following two claims show that $\sum b_i \geq l - g \log l$.

Claim 1: For each $0 \leq i < g$

$$|\alpha_i| \geq \frac{|\alpha_{i-1}|}{l2^{b_i}}$$

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$\alpha_x = \{r : (VP)(w,r) \text{ accepts via } s_{i-1}\#x\}$

Proof: Consider round i and the sets α_x defined in "obtain b_i". By definition the α_x's partition α_{i-1} and hence $\bigcup_x \alpha_x = \alpha_{i-1}$. Hence

$$|\bigcup \gamma_{\max}| \geq \frac{|\alpha_{i-1}|}{l}$$

(there are l possibilities for γ_i, thus at least one is of size total/l)

Since all members of $\gamma_{\max}$ differ in size by at most a factor of 2 and since $\alpha_i \in \gamma_{\max}$ we have

$$|\alpha_i| \geq \frac{|\bigcup \gamma_{\max}|}{2|\gamma_{\max}|}$$

and since $b_i = 2 + \lceil \log|\gamma_{\max}| \rceil$ we have

$$2^{b_{i+1}} \geq 2|\gamma_{\max}|$$

Thus

$$|\alpha_i| \geq \frac{|\bigcup \gamma_{\max}|}{2^{b_i}} \geq \frac{|\alpha_{i-1}|}{l 2^{b_i}}$$

■

(Claim 1/6)

(Full proof $w \in L$)

Claim 2: $\sum b_i \geq l - g \log l$

Proof: By Claim 1 we have:

$$|\alpha_g| \geq \frac{|\alpha_0|}{l^g \cdot \prod_{i \leq g} 2^{b_i}}$$

Since $|\alpha_0| \geq (2/3)2^l$ and taking logs

$$\log |\alpha_g| \geq (l-1) - (g \log l + \sum_{i \leq g} b_i)$$

Since $b_{g+1} > 1 + \log |\alpha_g|$

$$\sum_{i \leq g+1} b_i \geq l - g \log l$$

■

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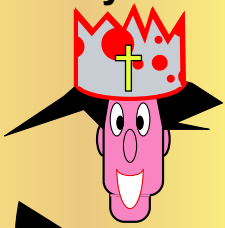

$w \notin L$

b_1

$H \in_R (\Sigma^m \to \Sigma^{b_1})^l,\ Z \in_R (\Sigma^m)^{l^2}$

$1 \le i \le g$

$x_l,\ y_l,\ b_{l+1}$

$x_i \in H^{-1}(Z)$?

$H \in_R (\Sigma^m \to \Sigma^{b_{i+1}})^l,\ Z \in_R (\Sigma^m)^{l^2}$

r

$r \in H^{-1}(Z)$?

$1 \le i \le g,\ \ V(w,r,s_{i-1}) = x_i$?

$V(w,r,s_g) =$ "accept" ?

$\sum b_i \ge l - g \log l$?

(←) Merlin's impotence when $w \notin W$

Show that if $\Pr[V(w) \text{ accepts}] \leq e$, then $\Pr[A(w) \text{ accepts}] \leq 1/3$.

For every $i > 0$ and $s_i = x_1 \# y_1 \# \cdots \# x_i \# y_i$ let $a(s_i) = \max_P \Pr[(V^* P)(w) \text{ accepts via } s_i]$. For each $x \in \Sigma^m$ let y_x be any $y \in \Sigma^m$ maximizing $a(s_i \# x \# y)$.

(Full proof $w \notin$ L)

The following three claims show that $a(s_{i+1})$ is likely to be much smaller than $a(s_i)$.

Claim 3: $a(s_i) = \sum_x a(s_i \#x\#y_x)$ ■

(Claim 3/6)

(Full proof $w \notin L$)

Fix $0 \le i < g$ and s_i. For every $c > 0$ let $X_c = \{x\colon a(s_i \# x \# y_x) \ge a(s_i)/c\}$

Claim 4: $|X_c| \le c$ ■

(Claim 4/6)

(Full proof $w \notin L$)

Fix $b,d>0$. Choose l random linear functions $H=\{h_1, \ldots, h_l\}$, $h_i:\Sigma^m \rightarrow \Sigma^b$ and l^2 random strings $Z \subseteq \Sigma^b$. Pick any $x \in H^{-1}(Z)$ and any $y \in \Sigma^m$. Let $s_{i+1}=s_i \# x \# y$.

We now describe a collection of events corresponding to exceptional luck on Merlin's part.

Call the following event E_{i+1}:

$$a(s_{i+1}) \geq \frac{a(s_i)}{2^b/d}$$

(Claim 5/6)

(Full proof $w \notin L$)

Claim 5: $\Pr[E_i] \le l^3/d$

Proof: Let $c = \lfloor d/2^b \rfloor$. Then $|X_c| \le 2^b/d$ by claim 4. Since $a(s_i \# x \# y_x) \ge a(s_{i+1})$ by the definition of y_x, if $a(s_{i+1}) \ge a(s_i)/(2^b/d)$ then $x \in X_c$. Since $x \in H^{-1}(Z)$,

$$Pr\left[a(s_{i+1}) \ge \frac{a(s_i)}{2^b/d}\right]$$

$$= \Pr[x \in X_c \cap H^{-1}(Z)]$$

$$= \Pr[H(X_c) \cap Z \ne 0]$$

$$\le l^3/d$$

by the approximate lower bound lemma part 2b. ■

if $|C| \le 2^b/d$, $d>0$, **then** $\Pr[H(C) \cap Z \ne \emptyset] \le l^3/d$ (Claim 5/6)

(Full proof $w \notin L$)

Fix s_g. Choose l random linear functions $H=\{h_1, \ldots, h_l\}$, $h_i:\Sigma^l \rightarrow \Sigma^{b_{g+1}}$ and l^2 random strings $Z \subseteq \Sigma^{b_{g+1}}$. Pick any $r \in H^{-1}(Z)$. Call the following event E_{g+1}:

$$2^l a(s_g) \leq 2^b/d \text{ and } (V^*P)(w,r) \text{ accepts via } s_g$$

Claim 6:

$$\Pr[E_{g+1}] \leq l^3/d$$

Proof: By the approximate lower bound lemma part 2b, since $|\{r: (V^*P)(w,r) \text{ accepts via } s_g\}| = 2^l a(s_g)$. ■

if $|C| \leq 2^b/d$, $d>0$, **then** $\Pr[H(C) \cap Z \neq \emptyset] \leq l^3/d$

(Claim 6/6)

(Full proof $w \notin L$)

In any run of A and M, event E_i may occur during round i, where $b = b_i$ for $i \leq g+1$. The probability that each occurs is at most l^3/d and therefore the probability that any occurs is at most $(g+1)l^3/d$. Choose

$$d = 3(g+1)l^3.$$

Then $\Pr[\exists\, i E_i \text{ occurs}] \leq 1/3$.

(Full proof $w \notin L$)

Assume no E_i occurs. Then we show that A will reject, provided that $\Pr[V(w)$ accepts$] \leq e$.

Since $\forall i \leq g$, $\neg E_i$, we have:

$$\frac{a(s_0)}{\prod_{i \leq g}(2^{b_i}/d)} \geq a(s_g)$$

Since $\neg E_{g+1}$:

$$(V^*P)(w,r) \neq \text{accept}$$

or

$$2^l a(s_g) \geq 2^{b_{g+1}}/d$$

(Full proof $w \notin L$)

Thus if $(V^*P)(w,r)$ accepts, combining the above:

$$2^l a(s_0) \geq \prod_{1 \leq i \leq g+1} (2^{b_i}/d)$$

so, since $l \geq g+1$, taking logs:

$$\begin{aligned} & l + \log a(s_0) \\ & \geq \sum b_i - (g+1)\log d \\ & \geq \sum b_i - (g+1) \\ & \geq \sum b_i - 10g\log l \end{aligned}$$

but

$$a(s_0) = \Pr[V(w) \text{ accepts}] \leq e \leq l^{-12g}$$

so

$$l - 12g^2 \log l \geq \sum b_i - 10g\log l$$

(Full proof $w \notin L$)

Thus

$$\sum b_i \leq l - 2g\log l < l - g\log l$$

Recall that Arthur only accepts if $(V^*P)(w,r)$ accepts and $\sum b_i \geq l - g\log l$. Therefore if $\forall\ i \leq g+1$, E_i occurs and $\Pr[V(w)$ accepts$] \leq e$, then Arthur will reject. Hence $\Pr[A(w)$ accepts$] \leq 1/3$. ■

(Full proof $w \notin L$)

Shafi Goldwasser, Michael Sipser

Private Coins versus Public Coins in Interactive Proof Systems