Pumping Lemma

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NON-Regular Languages

**THEOREM 1.70**

**Pumping lemma** If $A$ is a regular language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Application: any language that does not satisfy the pumping lemma is non-regular.

Note however that some non-regular languages DO satisfy the Pumping Lemma...
Pumping Lemma

\[ xyz = 101101110 \]
Pumping Lemma

\[ xyz = 101101110 \]
Pumping Lemma

\[ x_{yz} = 10110110 \]
Pumping Lemma

\[ x^{yz} = 101101110 \]
Pumping Lemma

\[ xyz = 101101110 \]
Pumping Lemma

\[ xyz = 101101110 \]
Pumping Lemma

\[ x y z = 101101110 \]
Pumping Lemma

\[ xy = 10110110 \]
Pumping Lemma

\[ xyz = 101101110 \]
Pumping Lemma

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Pumping Lemma

\[ xyz = 101101110 \]
Pumping Lemma

$\text{xz} = 101110$
Pumping Lemma

\[ xyyz = 101101101110 \]
Pumping Lemma

\[ xyyyyz = 101101101101110 \]
Pumping Lemma

If $|xyz| > \text{number-of-states}$ then $q_9$ exists...

**Figure 1.72**
Example showing how the strings $x$, $y$, and $z$ affect $M$
Pumping Lemma

**Proof:** Let $M$ be an automaton accepting $A$.

Let $n$ be the number of states of $M$.

Consider setting $p = n + 1$ as the pumping length. By the fact that $p > n$, any sequence of states $s_0 ... s_m$ accepting a string $w$ of length $m \geq p$ must contain two identical states $s_i = s_j$ with $j > i$. Let $j$ be the least index so that $s_j = s_i$ for some $i < j$ as above.
Pumping Lemma

Define $x$ to be the string digested by $M$ from $s_0$ to $s_i$, $y$ be the string digested by $M$ from $s_i$ to $s_j$ and $z$ be the string digested by $M$ from $s_j$ to $s_m$.

Since $j>i$ we have $|y|>0$ (2.).

Because our choice of $y$ produces a closed loop it is clear that zero, one, or many repetitions of $y$ will make no difference to being a member of $A$ or not (1.).
Define $x$ to be the string digested by $M$ from $s_0$ to $s_i$, $y$ be the string digested by $M$ from $s_i$ to $s_j$ and $z$ be the string digested by $M$ from $s_j$ to $s_m$.

We obtain:

$$S_0 \times_1 S_1 \times_2 S_2 \ldots S_{i-1} \times_i S_i Y_1 S_{i+1} Y_2 S_{i+2} \ldots S_{j-1} Y_{j-i} S_j Z_1 \ldots$$

where all states upto $s_{j-1}$ are distinct by the assumptions above. Thus $|xy| = i+j-i = j \leq p$ (3.).

QED
Theorem 1.70

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\[ A \in \text{REG} \implies \exists p \forall s \in A, |s| \geq p, \exists xyz = s \text{ st } 1, 2, 3 = \text{true}. \]

\[ \forall p \exists s \in A, |s| \geq p, \forall xyz = s \text{ [1 or 2 or 3 = false].} \]

\[ \implies A \notin \text{REG} \]
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\[ \forall p \exists s \in A, |s| \geq p, \forall xyz = s \text{ s.t. } 2, 3 = \text{true} \text{ [1=false]}. \]

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\[ \forall p \exists s \in A, \ |s| \geq p, \ \forall xyz = s \text{ s.t. } |y| > 0, |xy| < p, \] then \[ \exists i \geq 0 \text{ s.t. } s' = xy^iz \notin A. \]

\[ \implies A \notin \text{REG} \]
Application of the Pumping Lemma

\[ B = \{ 0^n1^n \mid n \geq 0 \} \text{ is NON-Regular.} \]

Assume \( B \) is regular. Then by the pumping Lemma there exists a pumping length \( p \) with properties 1., 2. and 3. satisfied. Take \( n=p \) and set \( s = 0^p1^p \in B \). Then by 3. \( xy \) contains only zeros. Therefore if we pump even once to obtain \( s' = xyyz = 0^q1^p \) it will contain more zeros than ones \((q>p)\) : a string \( s' \) not in \( B \). Thus \( B \) is non-regular.
∀p∃s∈B, |s|≥p, ∀xyz=s s.t. |y|>0,|xy|<p, then ∃i≥0 s.t. s′=xy^iz∉B.

⇒ B∉REG

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Application of the Pumping Lemma

\[ F = \{ \, ww \mid w \in \Sigma^* \, \} \text{ is NON-Regular.} \]

Assume \( F \) is regular. Then by the pumping Lemma there exists a pumping length \( p \) with properties 1., 2. and 3. satisfied. Take \( s = 0^p10^p1 \in F \). Then by 3. \( xy \) contains only zeros. Therefore if we pump even once to obtain \( s' = xyyz \) it will contain more zeros before the first one than after the first one: a string \( s' \) not in \( F \). Thus \( F \) is non-regular.
\[ \forall p \exists s \in F, \ |s| \geq p, \ \forall xyz = s \ \text{s.t.} \ |y| > 0, |xy| < p, \]
\[ \text{then } \exists i \geq 0 \ \text{s.t.} \ s' = xy^iz \not\in F. \]
\[ \Rightarrow F \not\in \text{REG} \]

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Application of the Pumping Lemma

E = \{ 0^i 1^j \mid i > j \geq 0 \} is NON-Regular.

Assume E is regular. Then by the pumping Lemma there exists a pumping length p with properties 1., 2. and 3. satisfied. Take i=p+1, j=p and obtain s=0^{p+1}1^p \in E. Then by 3. xy contains only zeros.
∀p∃s∈E, |s|≥p, ∀xyz=s s.t. |y|>0,|xy|<p, then ∃i≥0 s.t. s′=xyiz∉E.
⇒ E∉REG

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Assume E is regular. Then by the pumping Lemma there exists a pumping length p with properties 1., 2. and 3. satisfied. Take i=p+1, j=p and obtain s=0^{p+1}1^p∈E. Then by 3. xy contains only zeros.
Application of the Pumping Lemma

\[ E = \{ 0^i1^j \mid i > j \geq 0 \} \text{ is NON-Regular.} \]

Therefore if we pump up to obtain \( s' = xyyz = 0^k1^j \), \( k > i \) it will contain even more zeros than ones, which is still a string \( s' \) in \( E \). If we pump down however \( s'' = xz \), the number of zeros will become smaller or equal to the number of ones: an \( s'' \) not in \( E \). Thus \( E \) is non-regular.
∀p∃s∈E, |s|≥p, ∀xyz=s s.t. |y|>0,|xy|<p, then ∃i≥0 s.t. s''=xy^i z ∉ E.

⇒ E ∉ REG

E = \{ 0^i 1^j \mid i>j≥0 \} is NON-Regular.

Therefore if we pump up to obtain s'=xyyz=0^k 1^j , k>i it will contain even more zeros than ones, which is still a string s' in E. If we pump down however s''=xz, the number of zeros will become smaller or equal to the number of ones: an s'' not in E. Thus E is non-regular.
Application (?) of the Pumping Lemma

1.54 Consider the language $F = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$.

   a. Show that $F$ is not regular.

   b. Show that $F$ acts like a regular language in the pumping lemma. In other words, give a pumping length $p$ and demonstrate that $F$ satisfies the three conditions of the pumping lemma for this value of $p$.

   c. Explain why parts (a) and (b) do not contradict the pumping lemma.

   c. The Pumping Lemma says: if $A$ is regular then 1., 2. and 3. are satisfied. It does not say: if $A$ is not regular then 1., 2. or 3. is not satisfied... We can only conclude the opposite: if 1., 2. or 3. is not satisfied then $A$ is not regular...
Application of the Pumping Lemma

\[ D = \{ 1^{n^2} \mid n \geq 0 \} \text{ is NON-Regular.} \]

Assume \( D \) is regular. Then by the pumping Lemma there exists a pumping length \( p \) with properties 1., 2. and 3. satisfied. Take \( n=p \) and obtain \( s=1^{p^2} \).

Let \( i=|y| \leq p \). If we pump up we get \( s''=x^iyyz=1^{p^2+i} \).

Is it possible that both \( p^2 \) and \( p^2+i \) be perfect squares? No! The next square after \( p \) is \((p+1)^2 = p^2+2p+1 > p^2+p+1 > p^2+i\) proving that \( s'' \) is not in \( D \). So \( D \) is non-regular.
Application of the Pumping Lemma

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All languages

Computability Theory

Languages we can describe

Decidable Languages

Context-free Languages

Regular Languages

NON-Regular Languages
via Pumping Lemma

NON-Regular Languages
via Reductions
COMP-330
Theory of Computation
Fall 2019 -- Prof. Claude Crépeau
Lec. 9 :
the Pumping Lemma