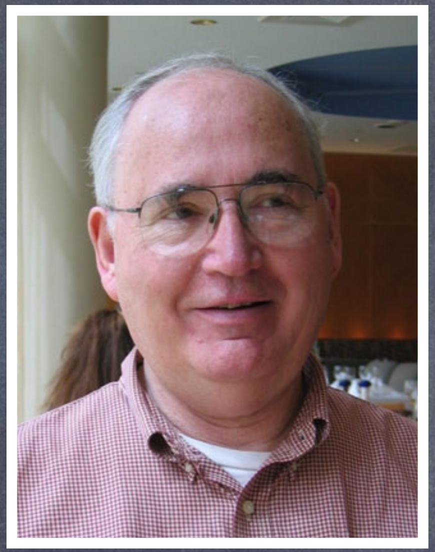
COMP-330
Theory of Computation

Fall 2019 -- Prof. Claude Crépeau

Lec. 9 :
the Pumping Lemma

Pumping Lemma

Michael Rabin



Dana Scott

NON-Regular Languages

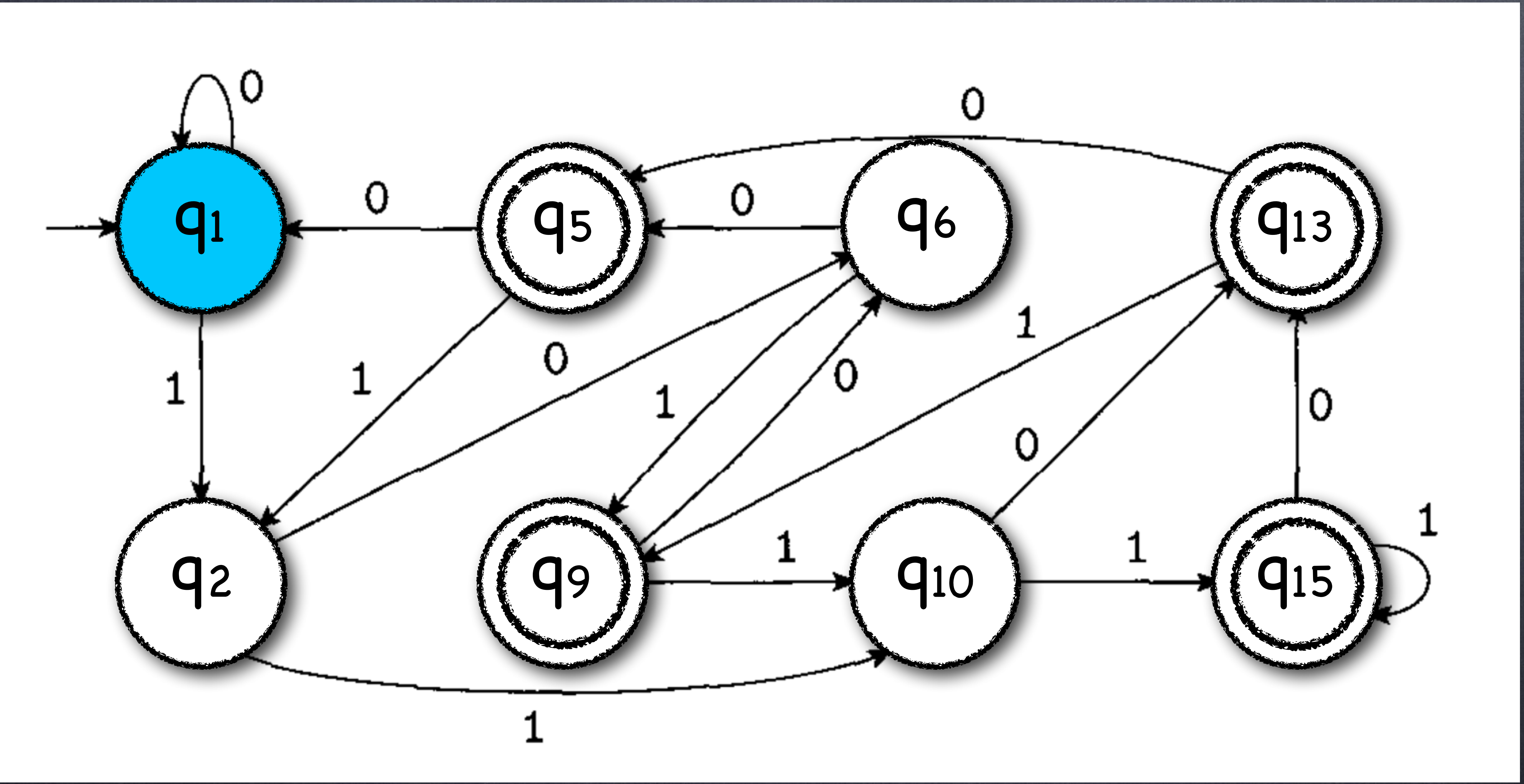
THEOREM 1.70

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

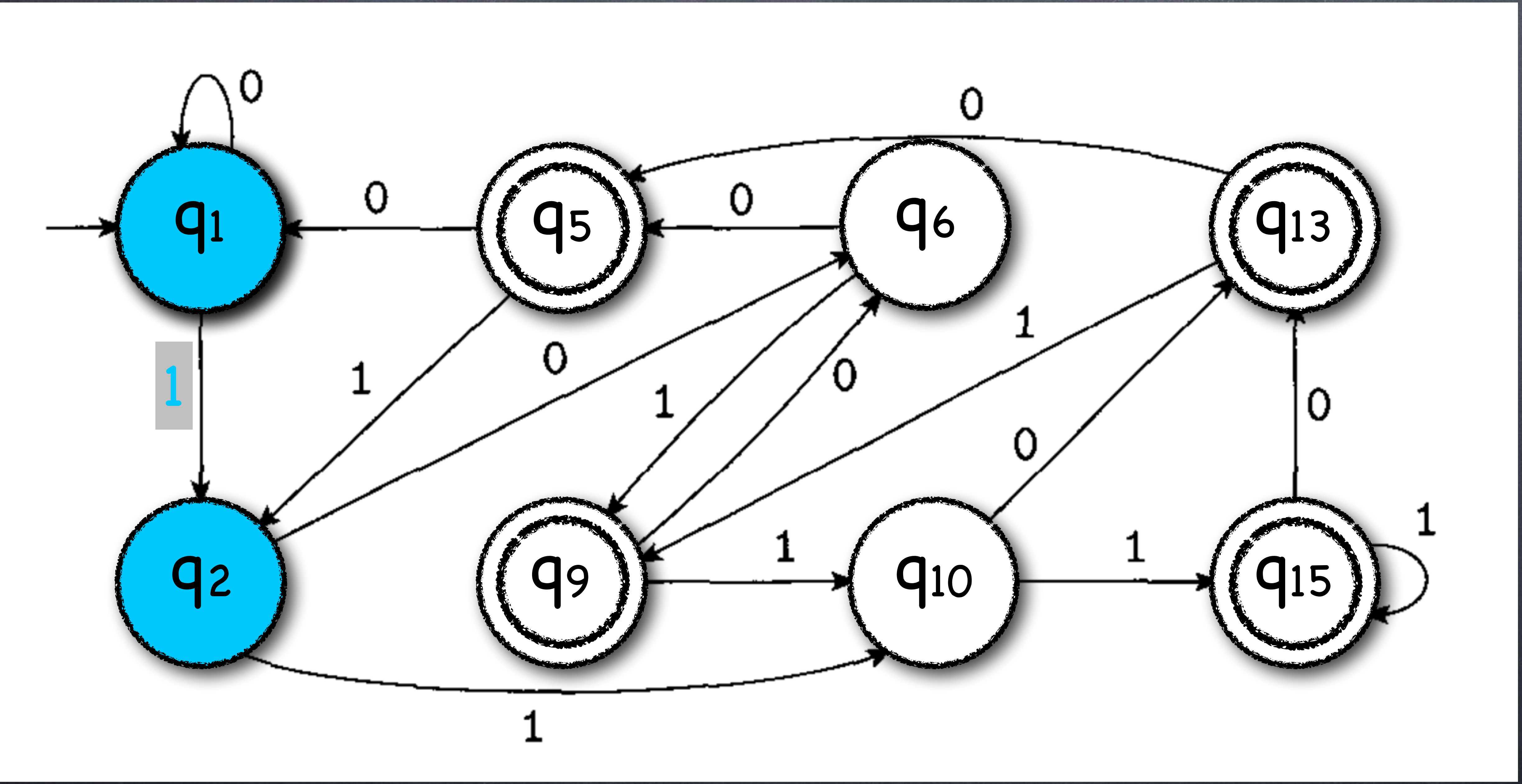
- Application: any language that does not satisfy the pumping lemma is non-regular.
- Note however that some non-regular languages DO satisfy the Pumping Lemma...

Pumping Lemma



xyz = 101101110

Pumping Lemma



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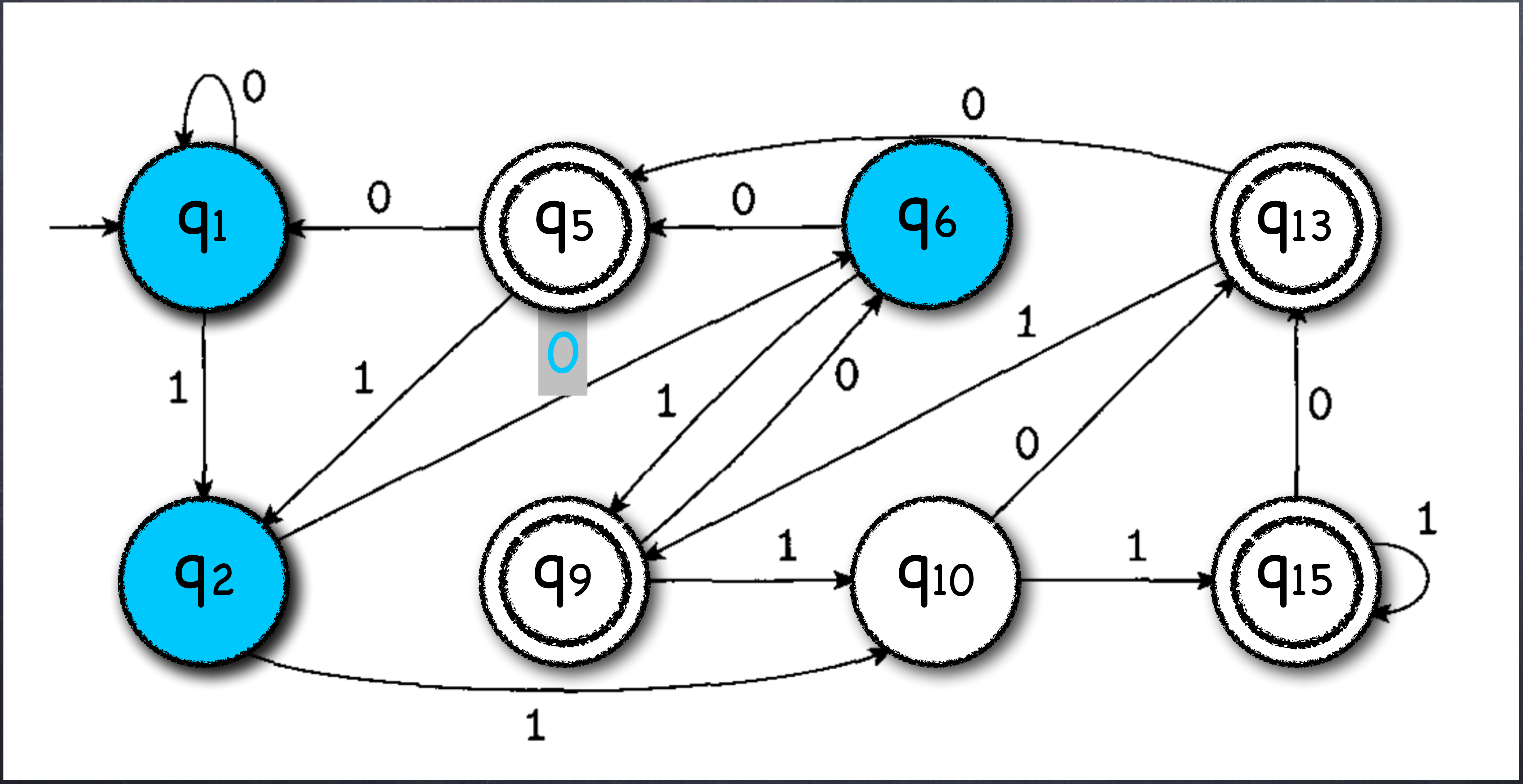

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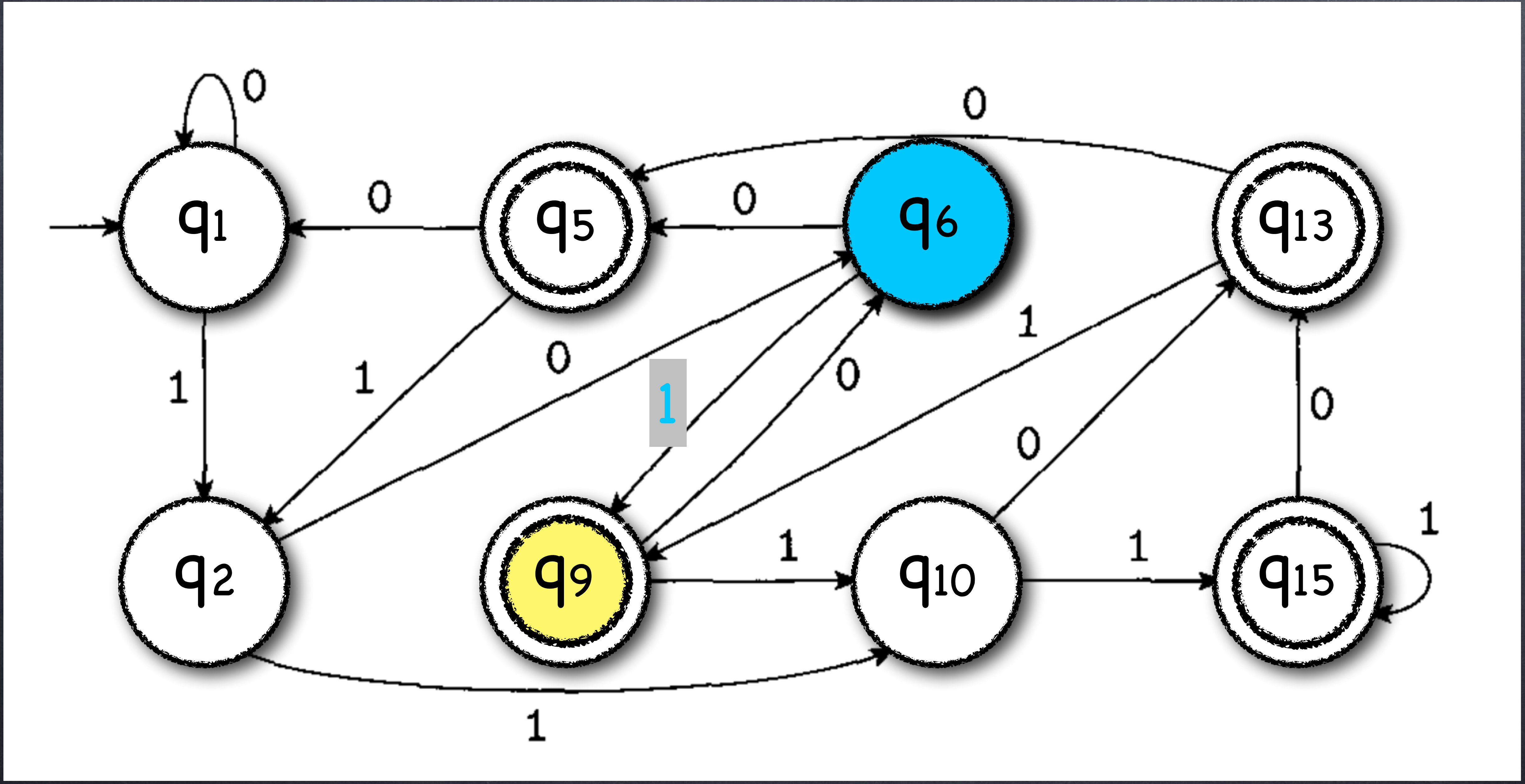

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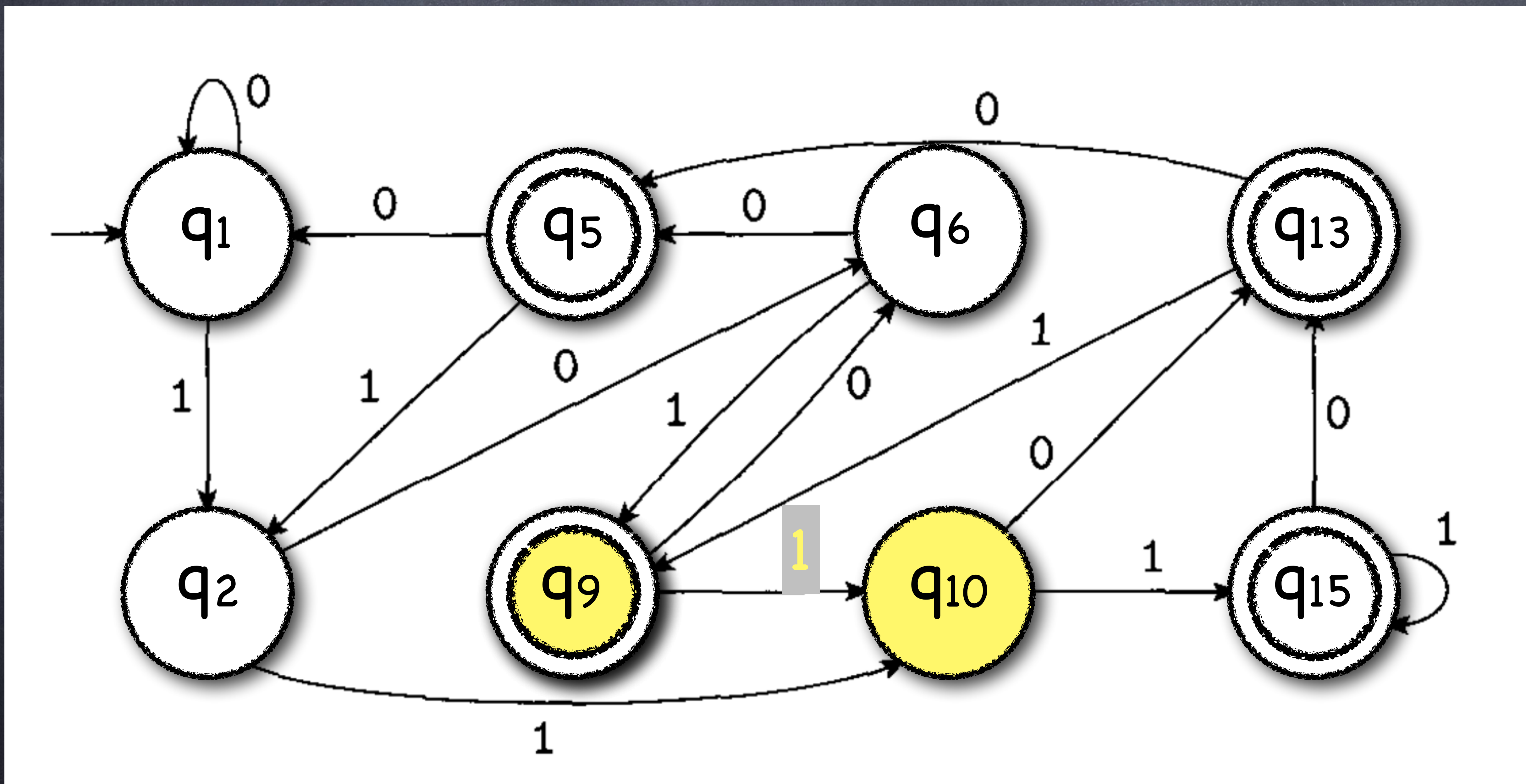

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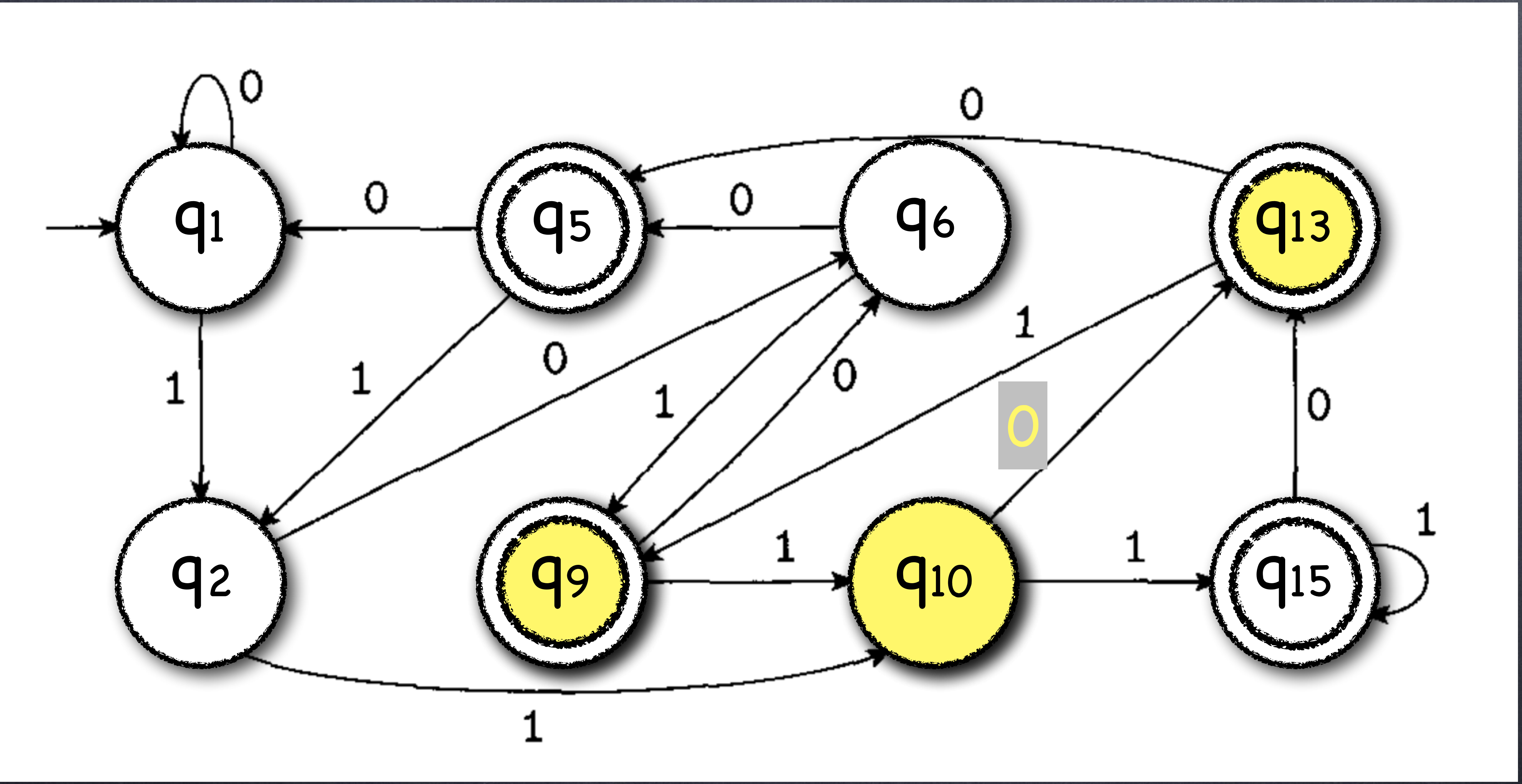

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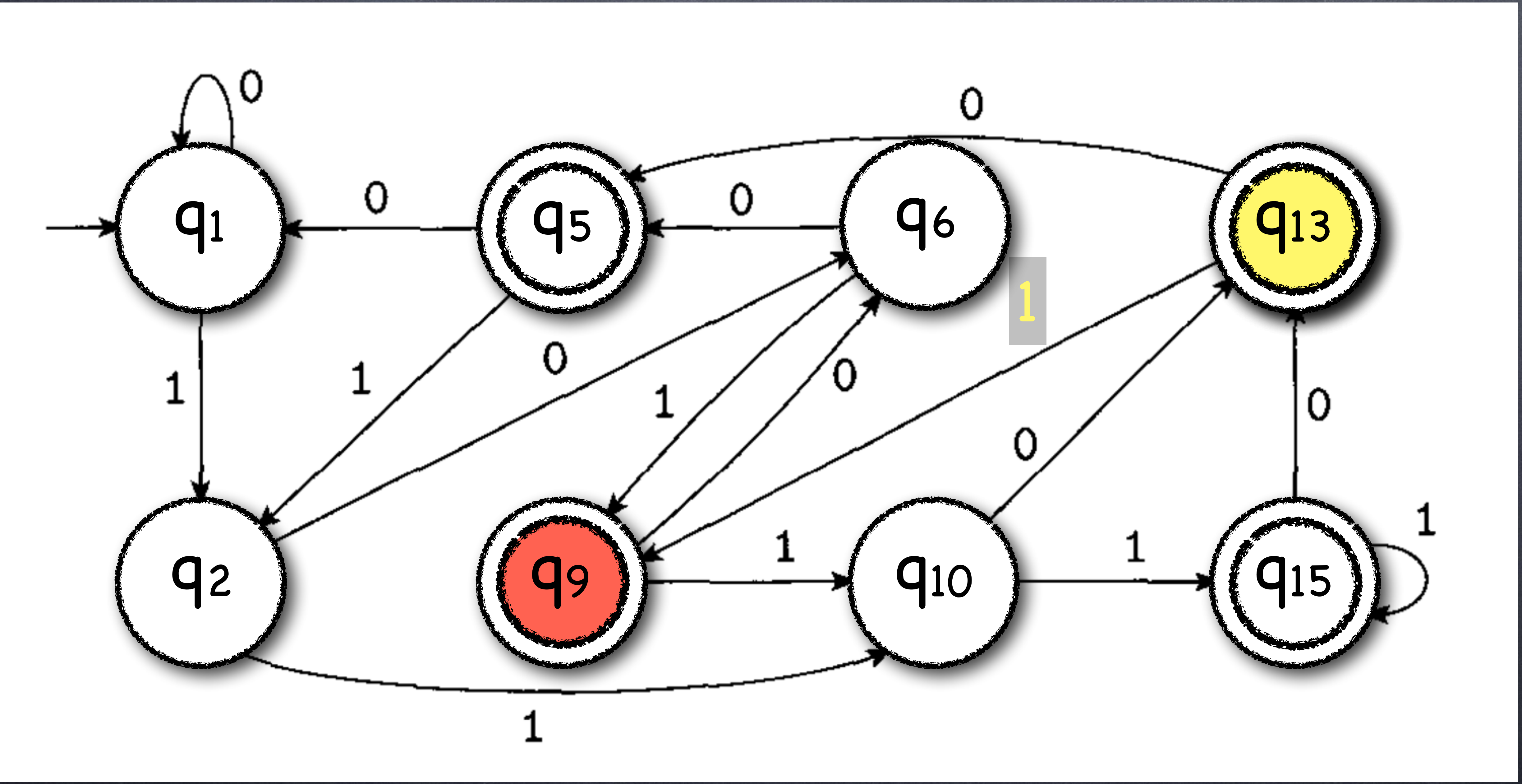

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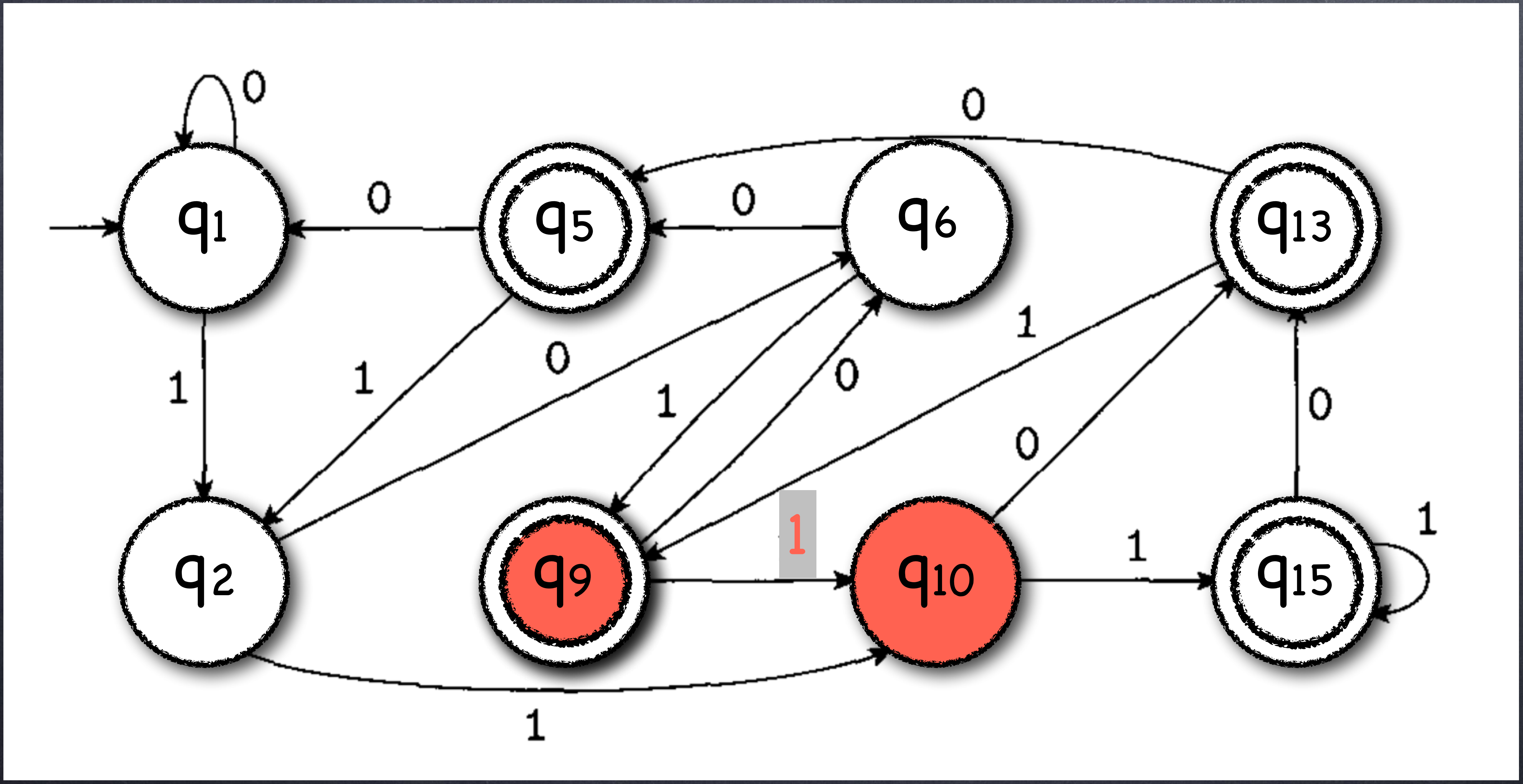

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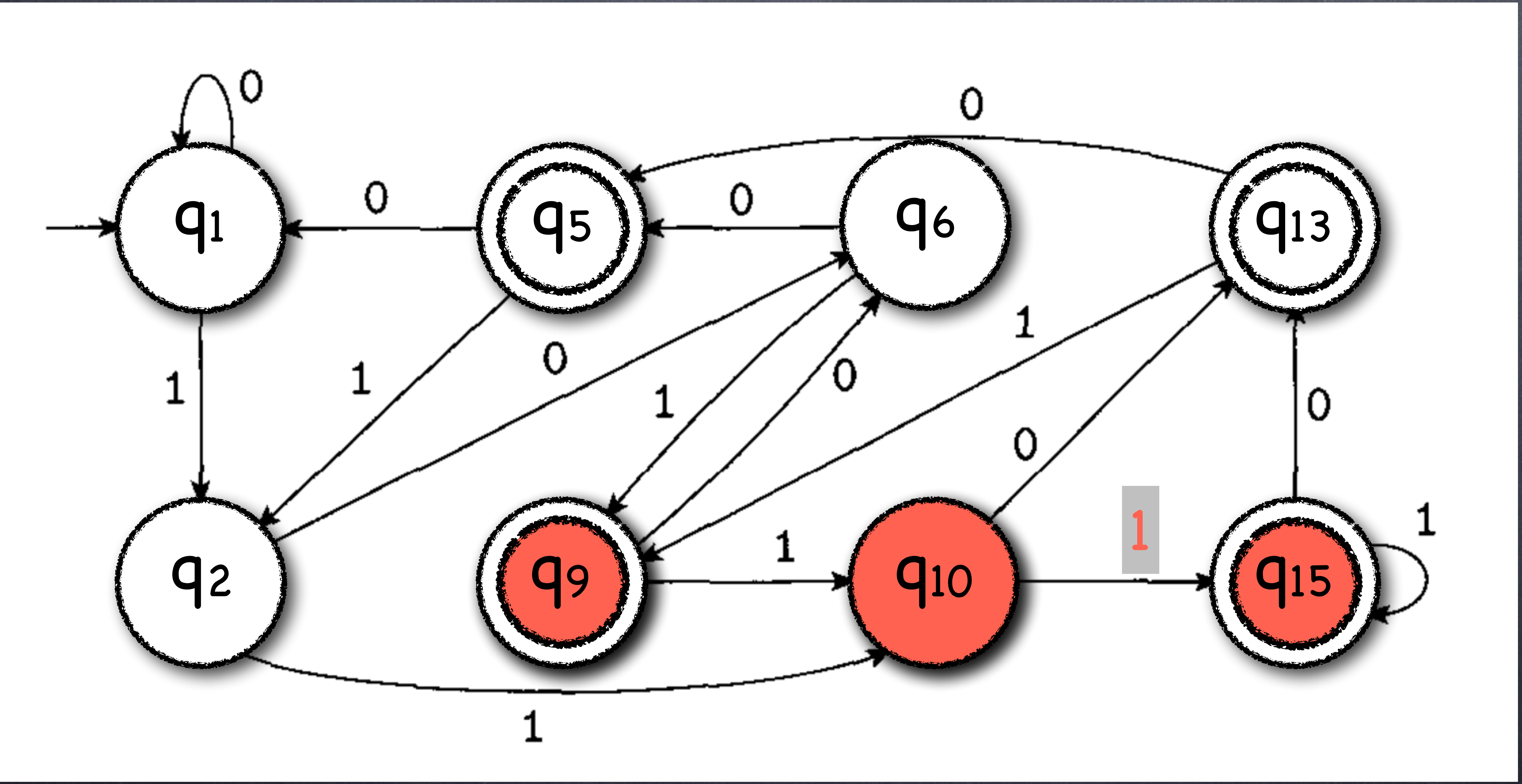

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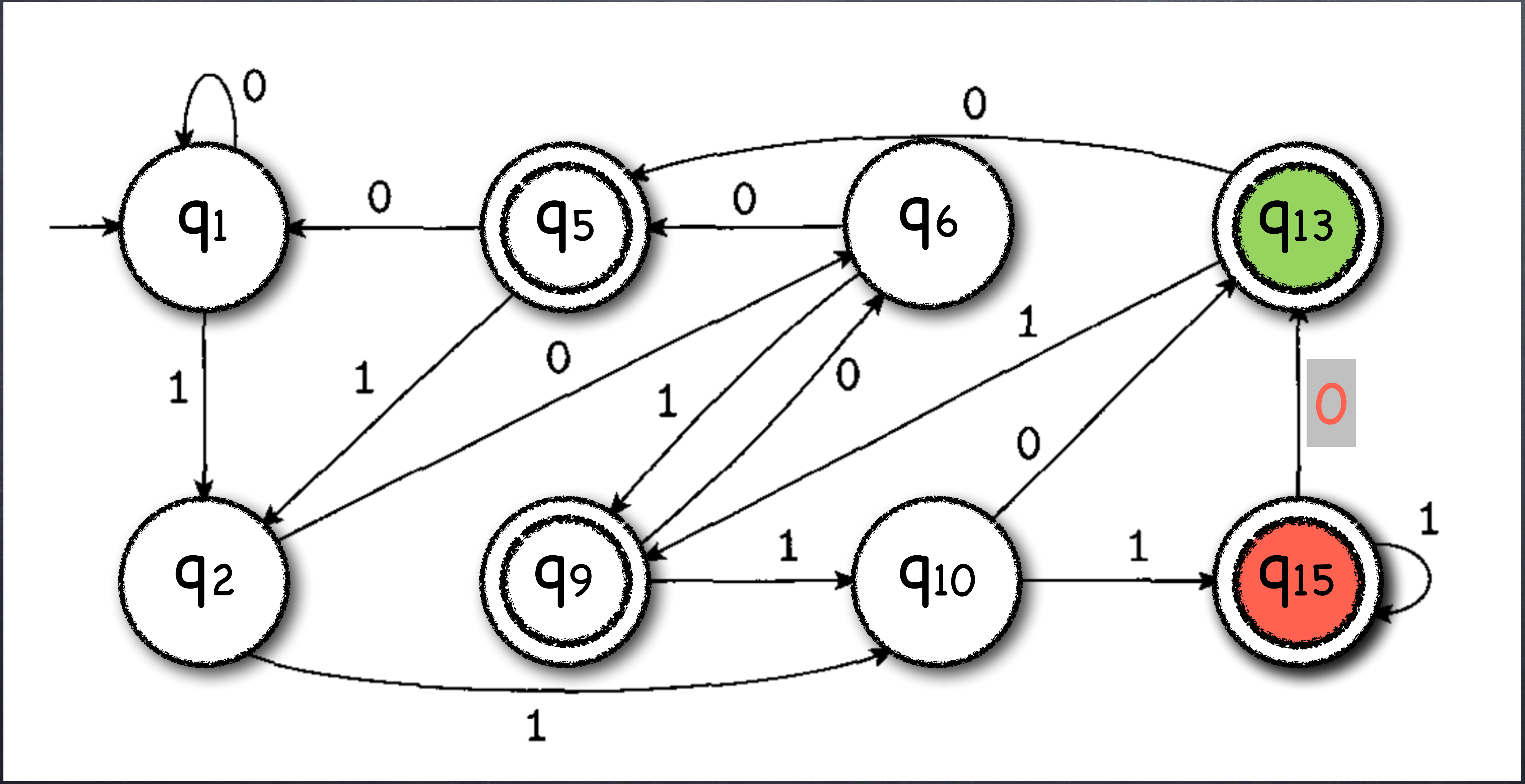

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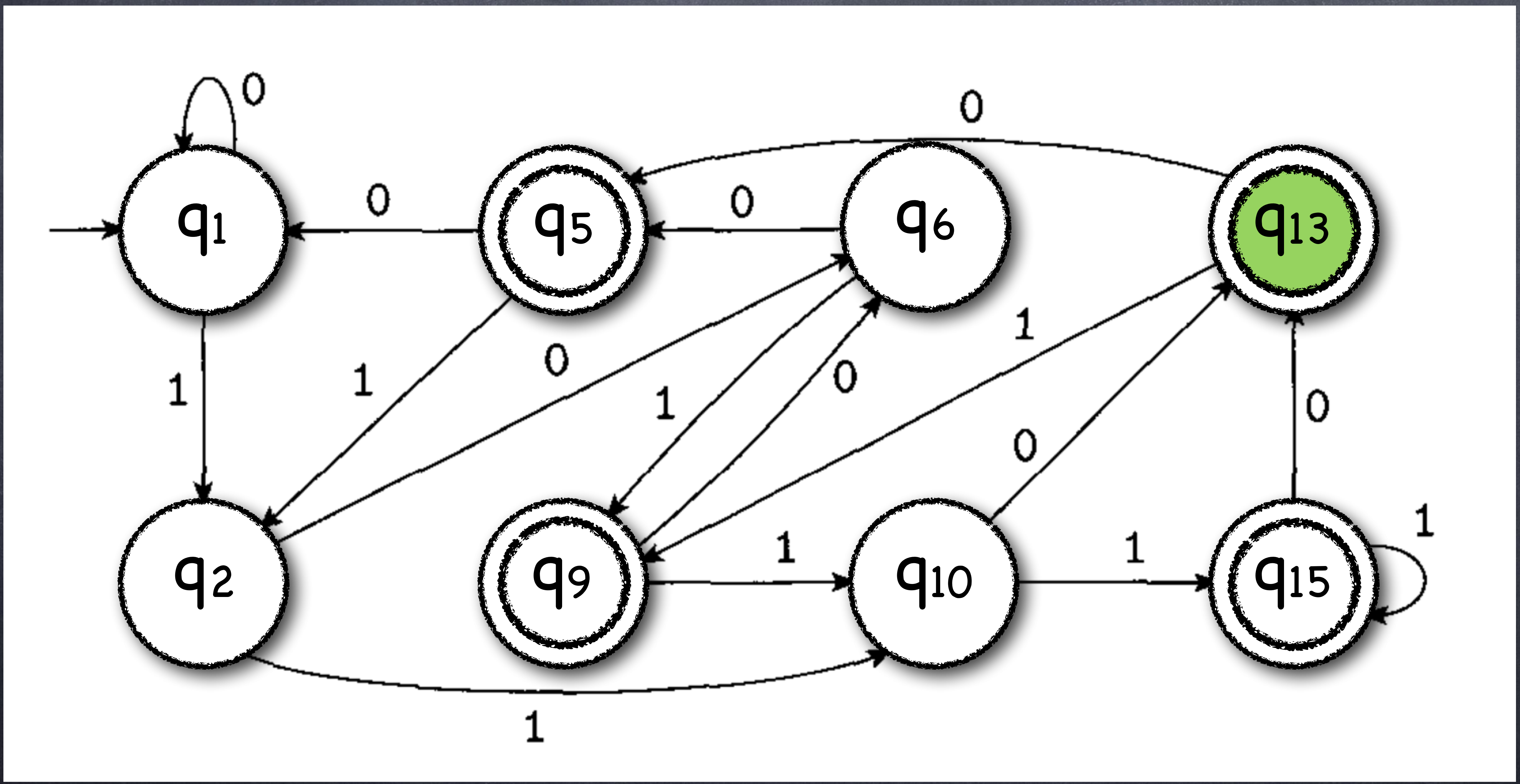

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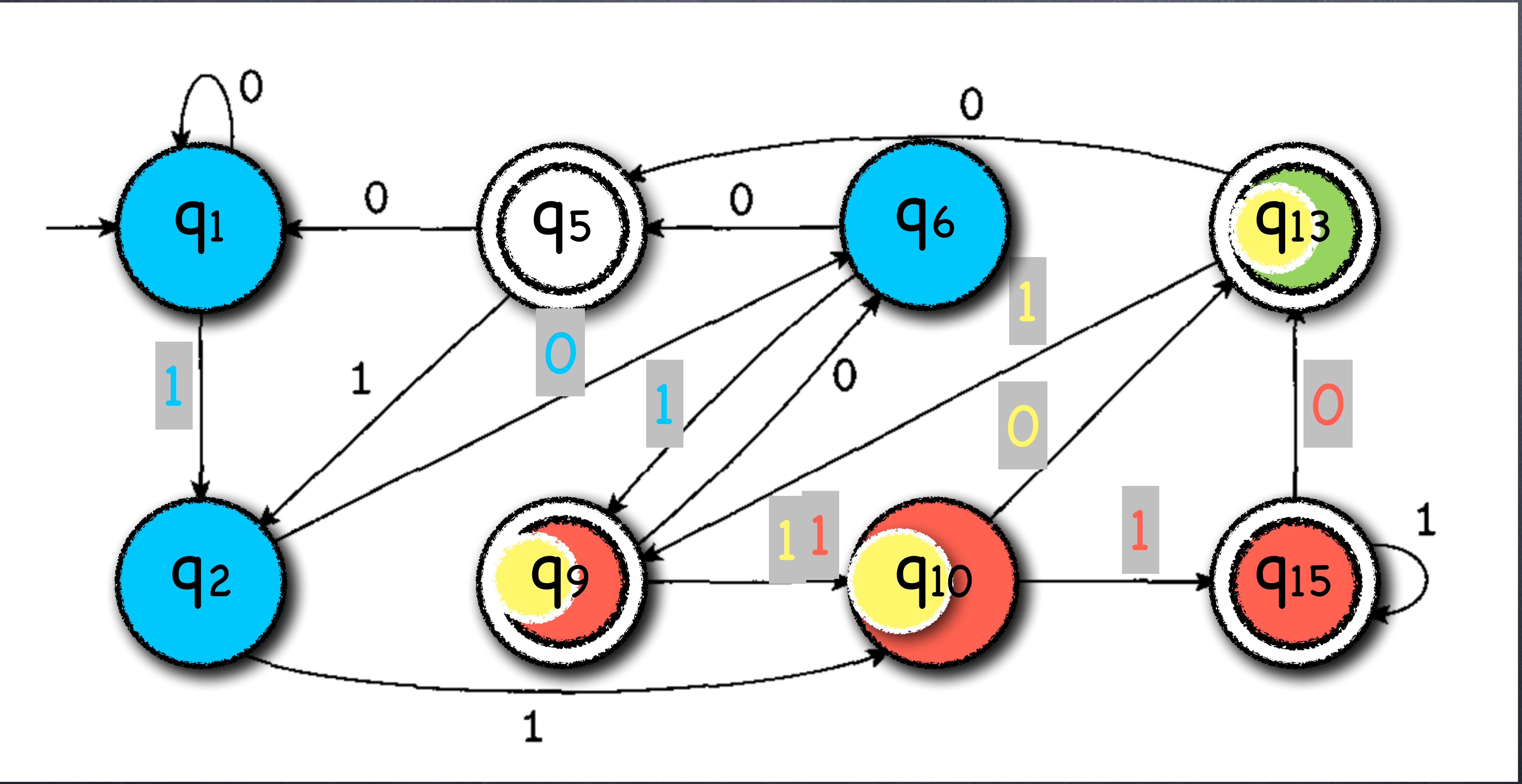

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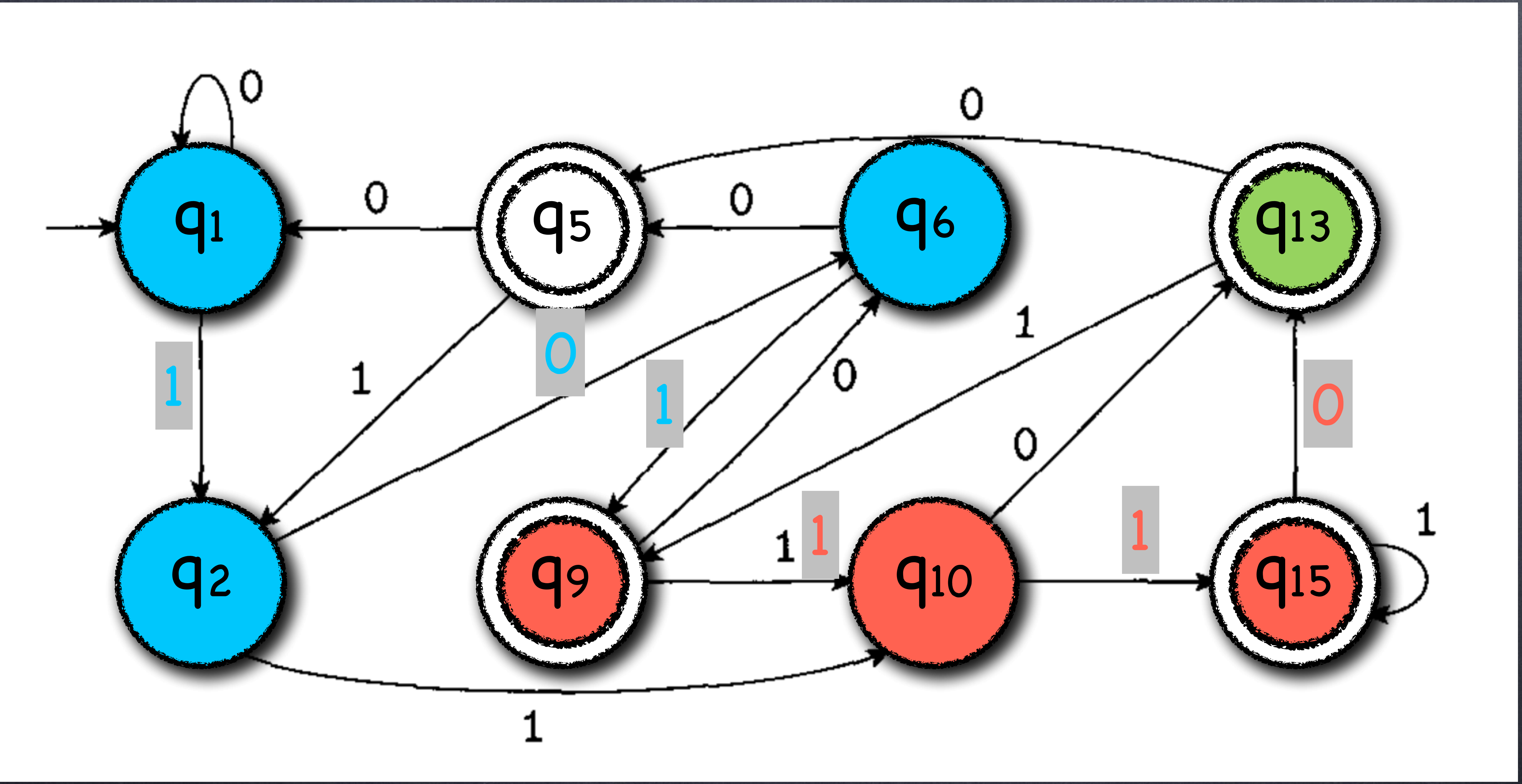

xz = 101110

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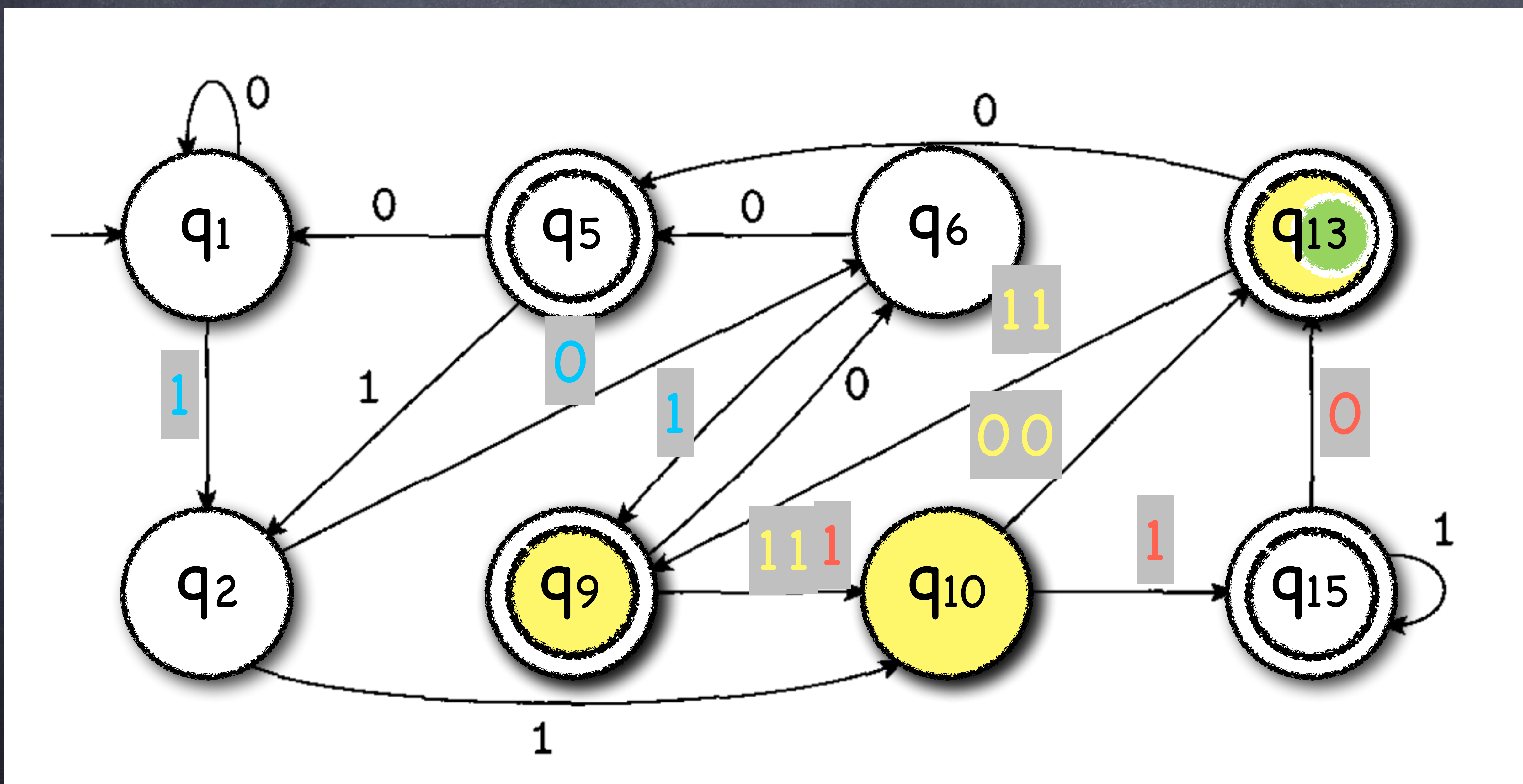

xyyz = 101101101110

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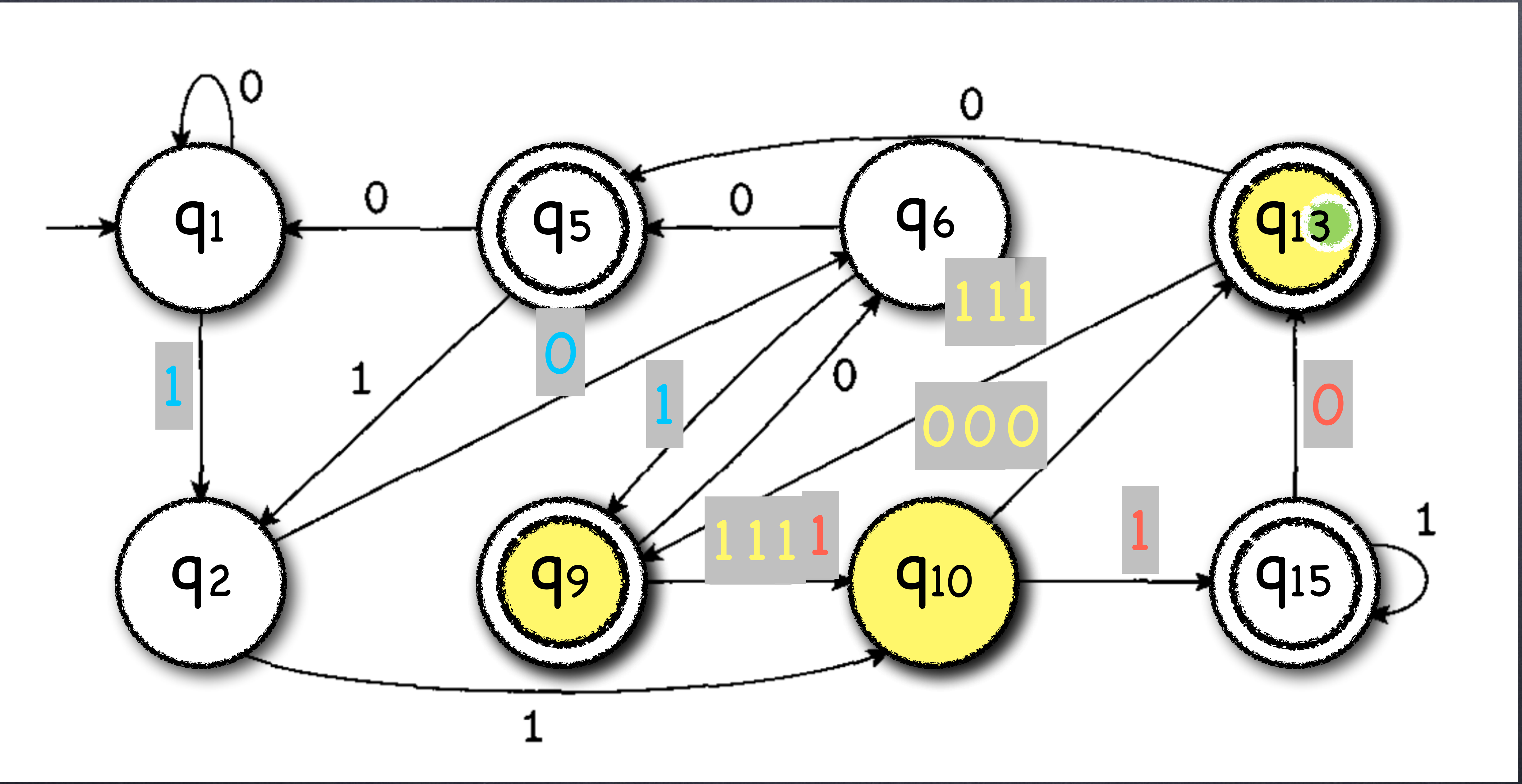

xyyyz = 101101101101110

Pumping Lemma

- If |xyz|>number-of-states then q_9 exists...

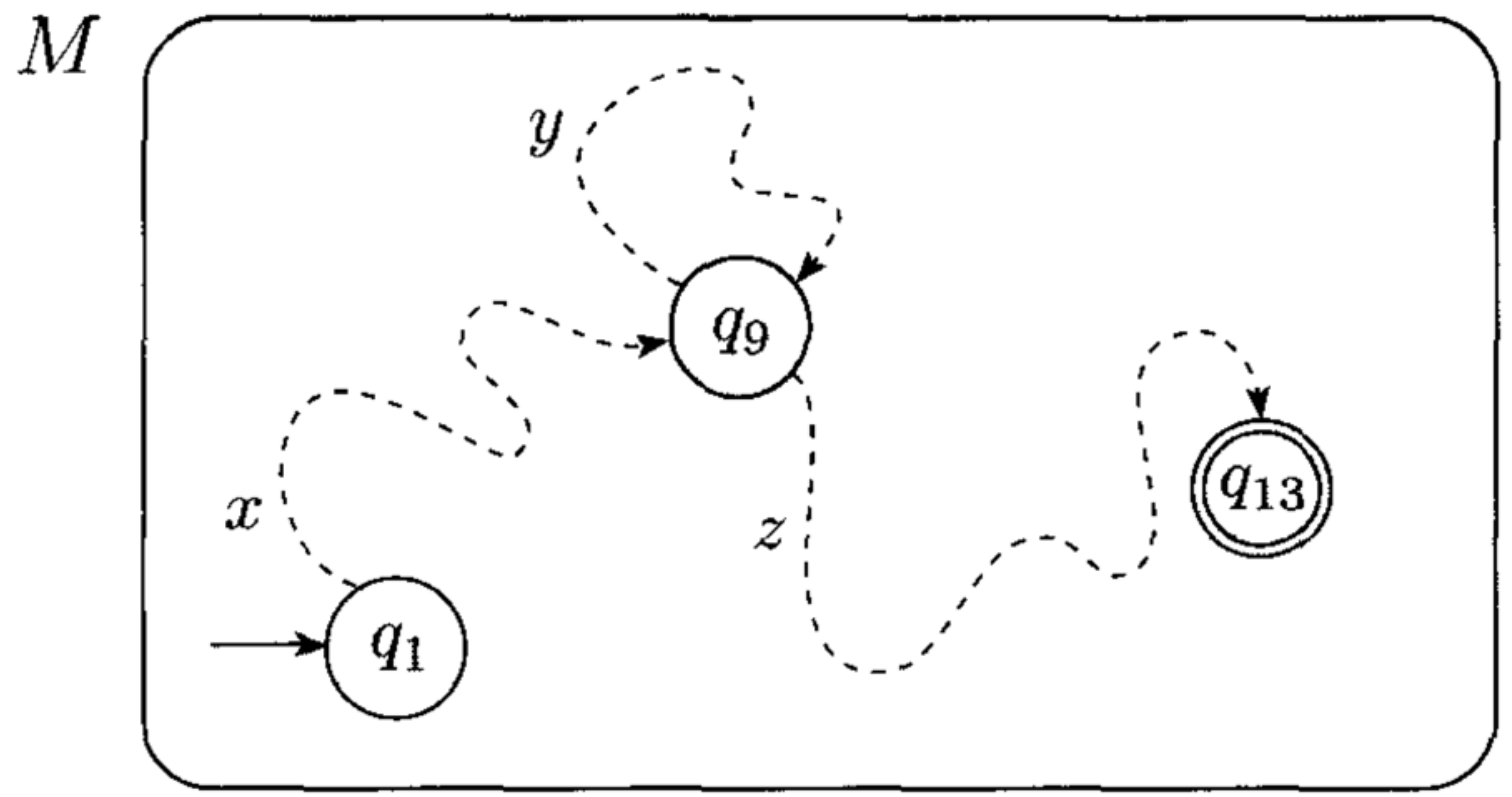

FIGURE 1.72
Example showing how the strings x, y, and z affect M

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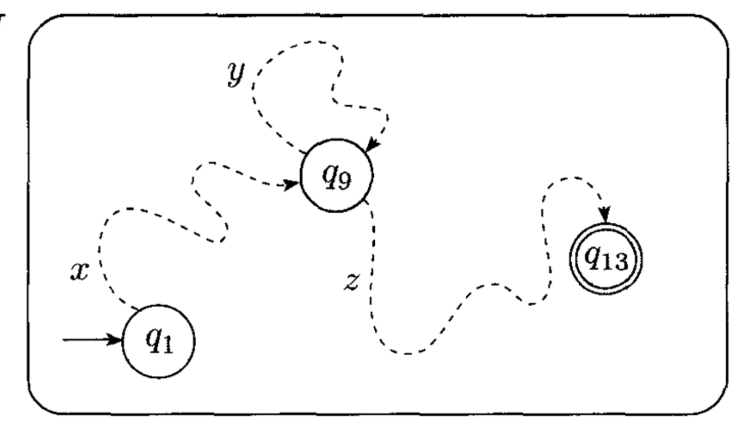


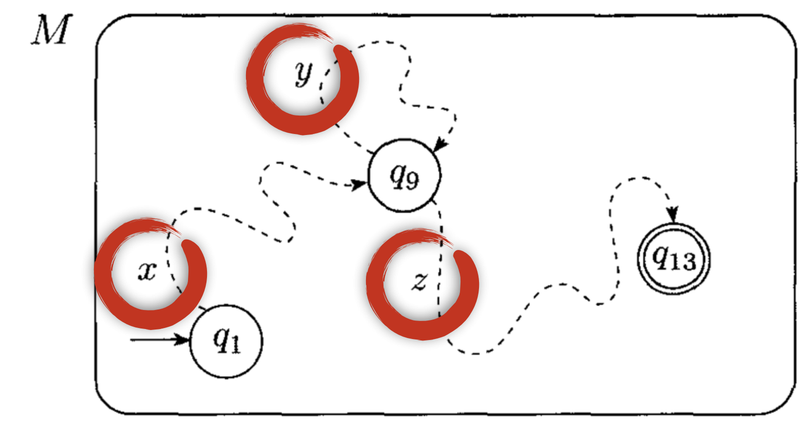
FIGURE 1.72
Example showing how the strings x, y, and z affect M

- **Proof**: Let M be an automaton accepting A.

  Let n be the number of states of M.

  Consider setting p=n+1 as the pumping length. By the fact that p>n, any sequence of states $s_0 \ldots s_m$ accepting a string w of length $m \geq p$ must contain two identical states $s_i = s_j$ with j>i. Let j be the least index so that $s_j = s_i$ for some i<j as above.

Pumping Lemma



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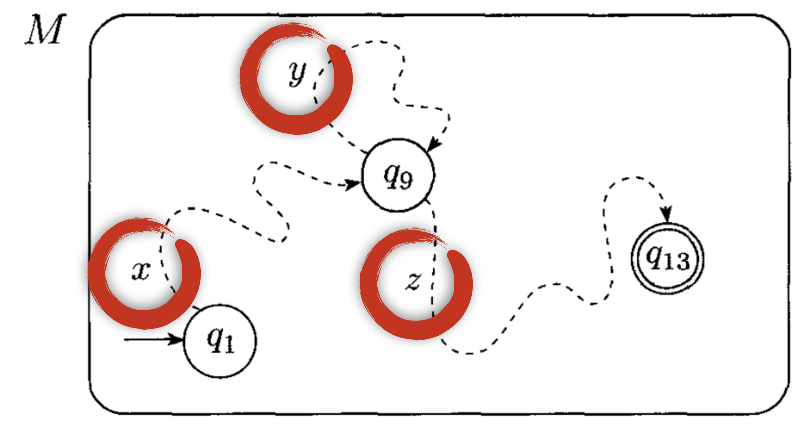
Example showing how the strings x, y, and z affect M

- Define x to be the string digested by M from s_0 to s_i, y be the string digested by M from s_i to s_j and z be the string digested by M from s_j to s_m.
- Since $j>i$ we have $|y|>0$ (2.).
- Because our choice of y produces a closed loop it is clear that zero, one, or many repetitions of y will make no difference to being a member of A or not (1.).

Pumping Lemma



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- Define x to be the string digested by M from s_0 to s_i, y be the string digested by M from s_i to s_j and z be the string digested by M from s_j to s_m.

- We obtain:

$s_0\ x_1\ s_1\ x_2\ s_2 \ldots s_{i-1}\ x_i\ s_i\ y_1\ s_{i+1}\ y_2\ s_{i+2} \ldots s_{j-1}\ y_{j-i}\ s_j\ z_1 \ldots$

where all states upto s_{j-1} are disctinct by the assumptions above. Thus $|xy| = i+j-i = j \leq p$ (3.).

QED

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$\forall p \exists s \in A, |s| \geq p, \forall xyz = s$ s.t. $|y| > 0, |xy| < p$,

then $\exists i \geq 0$ s.t. $s' = xy^iz \notin A$.

$\implies A \notin \mathbb{REG}$

Application of the Pumping Lemma

- $B = \{ 0^n1^n \mid n \geq 0 \}$ is NON-Regular.
- Assume B is regular. Then by the pumping Lemma there exists a pumping length p with properties 1., 2. and 3. satisfied. Take n=p and set $s = 0^p1^p \in B$. Then by 3. xy contains only zeros. Therefore if we pump even once to obtain $s' = xyyz = 0^q1^p$ it will contain more zeros than ones $(q>p)$: a string s' not in B. Thus B is non-regular.

$\forall p \exists s \in B, |s| \geq p, \forall xyz=s$ s.t. $|y|>0, |xy|<p$,
then $\exists i \geq 0$ s.t. $s'=xy^iz \notin B$.
$\Longrightarrow B \notin \mathbb{REG}$

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Application of the Pumping Lemma

- $F = \{ ww \mid w \in \Sigma^* \}$ is NON-Regular.
- Assume F is regular. Then by the pumping Lemma there exists a pumping length p with properties 1., 2. and 3. satisfied. Take $s=0^p10^p1 \in F$. Then by 3. xy contains only zeros. Therefore if we pump even once to obtain $s'=xyyz$ it will contain more zeros before the first one than after the first one : a string s' not in F. Thus F is non-regular.

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Application of the Pumping Lemma

- $E = \{ 0^i 1^j \mid i > j \geq 0 \}$ is NON-Regular.
- Assume E is regular. Then by the pumping Lemma there exists a pumping length p with properties 1., 2. and 3. satisfied. Take i=p+1, j=p and obtain $s = 0^{p+1} 1^p \in E$. Then by 3. xy contains only zeros.

$\forall p \exists s \in E, |s| \geq p, \forall xyz=s$ s.t. $|y|>0, |xy|<p$,
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Application of the Pumping Lemma

- $E = \{ 0^i1^j \mid i>j\geq0 \}$ is NON-Regular.
- Therefore if we pump up to obtain $s'=xyyz=0^k1^j$, $k>i$ it will contain even more zeros than ones, which is still a string s' in E. If we pump down however s"=xz, the number of zeros will become smaller or equal to the number of ones: an s" not in E. Thus E is non-regular.

$\forall p \exists s \in E, |s| \geq p, \forall xyz=s$ s.t. $|y|>0, |xy|<p$,
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Application (?) of the Pumping Lemma

1.54 Consider the language $F = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$.

a. Show that F is not regular.

b. Show that F acts like a regular language in the pumping lemma. In other words, give a pumping length p and demonstrate that F satisfies the three conditions of the pumping lemma for this value of p.

c. Explain why parts (a) and (b) do not contradict the pumping lemma.

- c. The Pumping Lemma says: if A is regular then 1., 2. and 3. are satisfied. It does not say: if A is not regular then 1., 2. or 3. is not satisfied... We can only conclude the opposite: if 1., 2. or 3. is not satisfied then A is not regular...

Application of the Pumping Lemma

- $D = \{ 1^{n^2} \mid n \geq 0 \}$ is NON-Regular.

- Assume D is regular. Then by the pumping Lemma there exists a pumping length p with properties 1., 2. and 3. satisfied. Take n=p and obtain $s=1^{p^2}$. Let $i=|y| \leq p$. If we pump up we get $s''=xyyz=1^{p^2+i}$. Is it possible that both p^2 and p^2+i be perfect squares ? No! The next square after p is

$$(p+1)^2 = p^2+2p+1 > p^2+p+1 > p^2+i$$

proving that s" is not in D. So D is non-regular.

$\forall p \exists s \in D, |s| \geq p, \forall xyz=s$ s.t. $|y|>0, |xy|<p$,
then $\exists i \geq 0$ s.t. $s'' = xy^i z \notin D$.
$\Longrightarrow D \notin \mathbb{REG}$

- $D = \{ 1^{n^2} \mid n \geq 0 \}$ is NON-Regular.

- Assume D is regular. Then by the pumping Lemma there exists a pumping length p with properties 1., 2. and 3. satisfied. Take n=p and obtain $s=1^{p^2}$. Let $i=|y| \leq p$. If we pump up we get $s''=xyyz=1^{p^2+i}$. Is it possible that both p^2 and p^2+i be perfect squares ? No! The next square after p is

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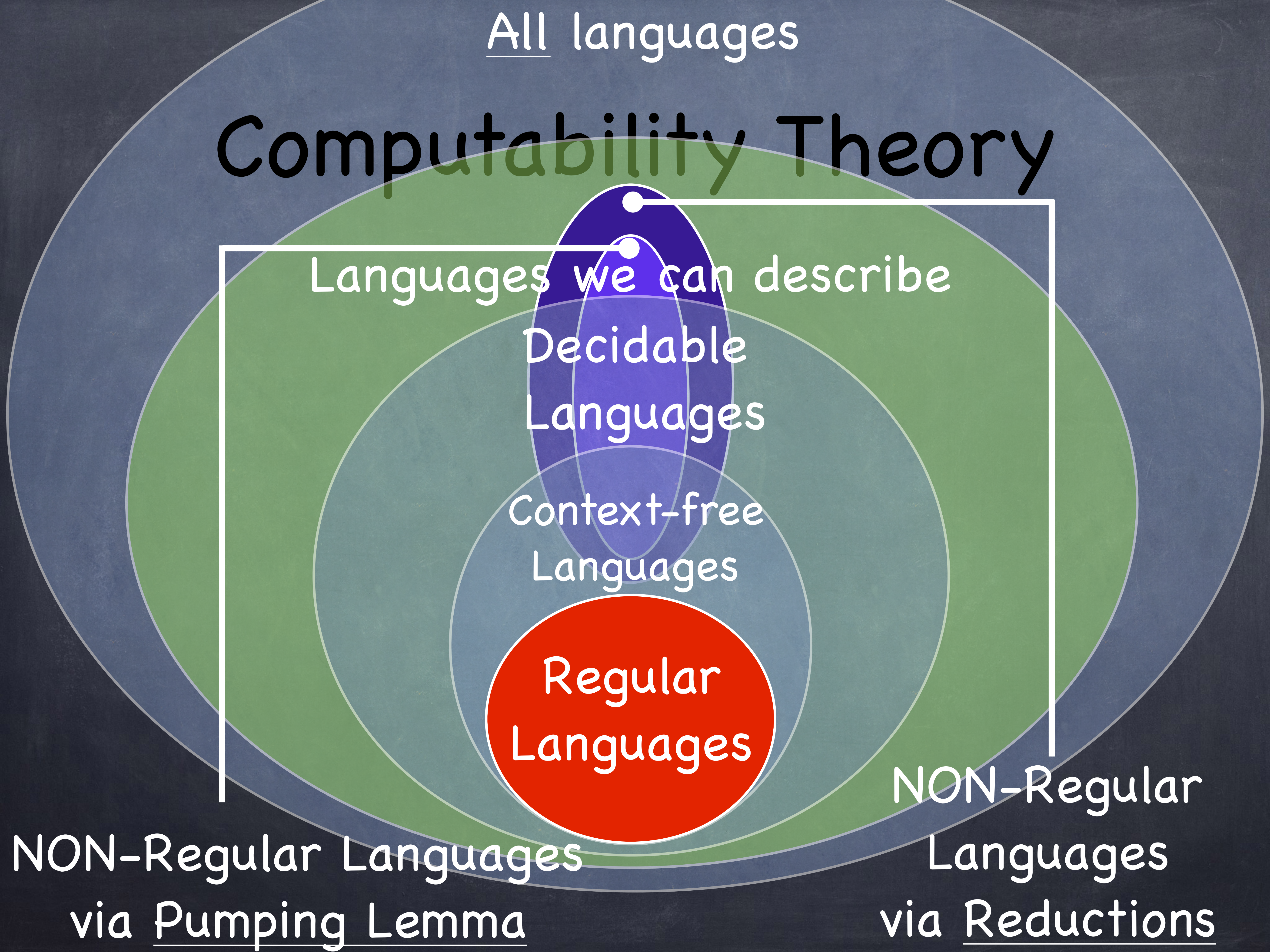
All languages
Computability Theory
Languages we can describe
Decidable Languages
Context-free Languages
Regular Languages
NON-Regular Languages via Pumping Lemma
NON-Regular Languages via Reductions

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