COMP-330
Theory of Computation

Fall 2019 -- Prof. Claude Crépeau

Lec. 5 : NFA-DFA equivalence
Definition of NFA

Let $N = (Q, \Sigma, \delta, q_0, F)$ be a nondeterministic finite state automaton and let $w = w_1 w_2 \ldots w_n$ ($n \geq 0$) be a string where each symbol $w_i \in \Sigma$.

$N$ accepts $w$ if $\exists m \geq n$, $\exists s_0, s_1, \ldots, s_m$ and $\exists y_1 y_2 \ldots y_m = w$, with each $y_i \in \Sigma \epsilon$ s.t.

1. $s_0 = q_0$
2. $s_{i+1} \in \delta(s_i, y_{i+1})$ for $i = 0 \ldots m-1$, and
3. $s_m \in F$
NFA-DFA equivalence
Theorem 1.39

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

Corollary 1.40

A language is regular if and only if some nondeterministic finite automaton recognizes it.
NFA-DFA equivalence
(without empty transitions)
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(without empty transitions)

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA (without empty transitions) accepting language $A$. We show a DFA $M = (Q', \Sigma, \delta', q'_0, F')$ accepting $A$. 
NFA–DFA equivalence
(without empty transitions)

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA (without empty transitions) accepting language $\mathcal{A}$. We show a DFA $M = (Q', \Sigma, \delta', q'_0, F')$ accepting $\mathcal{A}$.

$Q' = \mathcal{P}(Q) = \{ R \mid R \subseteq Q \}$
NFA-DFA equivalence

(without empty transitions)

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$\delta'(R,a) = \{ q \in Q \mid \exists r \in R, q \in \delta(r,a) \}$
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- $Q' = \mathcal{P}(Q) = \{ R \mid R \subseteq Q \}$
- $\delta'(R, a) = \{ q \in Q \mid \exists r \in R, \ q \in \delta(r, a) \}$
- $q'_0 = \{q_0\}$
- $F' = \{ R \in Q' \mid R \cap F \neq \emptyset \}$
\[ N_{2} \]

- \( q_{0000} = q_{\emptyset} \)
- \( q_{0001} = q_{\{1\}} \)
- \( q_{0011} = q_{\{1,2\}} \)
- \( q_{0010} = q_{\{2\}} \)
- \( q_{0101} = q_{\{1,3\}} \)
- \( q_{0100} = q_{\{3\}} \)
- \( q_{0110} = q_{\{2\}} \)
- \( q_{0111} = q_{\{2,3\}} \)
- \( q_{1001} = q_{\{1,4\}} \)
- \( q_{1000} = q_{\{4\}} \)
- \( q_{1010} = q_{\{2,4\}} \)
- \( q_{1011} = q_{\{1,2,4\}} \)
- \( q_{1101} = q_{\{1,3,4\}} \)
- \( q_{1100} = q_{\{3,4\}} \)
- \( q_{1110} = q_{\{2,3,4\}} \)
- \( q_{1111} = q_{\{1,2,3,4\}} \)

\[ q_{w_4 w_3 w_2 w_1} = q_{R} : (w_i = 1 \iff i \in R) \]
NFA-DFA equivalence
(with empty transitions)

\[ E(R) = \{ q \mid q \text{ can be reached from } R \text{ by traveling along } 0 \text{ or more } \varepsilon \text{ arrows} \}. \]
NFA-DFA equivalence
(with empty transitions)

Let \( N = (Q, \Sigma, \delta, q_0, F) \) be an NFA accepting language \( A \). We construct a DFA \( M = (Q', \Sigma, \delta', q_0', F') \) accepting \( A \) as well.

\[ E(R) = \{ q \mid q \text{ can be reached from } R \text{ by traveling along } 0 \text{ or more } \varepsilon \text{ arrows} \}. \]
NFA-DFA equivalence  
(with empty transitions)

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA accepting language $A$. We construct a DFA $M = (Q', \Sigma, \delta', q'_0, F')$ accepting $A$ as well.

$Q' = \mathcal{P}(Q) = \{ R \mid R \subseteq Q \}$

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$Q' = \mathcal{P}(Q) = \{ R \mid R \subseteq Q \}$

$E(R) = \{ q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon \text{ arrows} \}.$

$\delta'(R,a) = \{ q \in Q \mid \exists r \in R, q \in E(\delta(r,a)) \}, \forall a \neq \varepsilon$
NFA-DFA equivalence
(with empty transitions)

Let $N = (Q, Σ, δ, q_0, F)$ be an NFA accepting language $A$. We construct a DFA $M = (Q', Σ, δ', q'_0, F')$ accepting $A$ as well.

$Q' = \mathcal{P}(Q) = \{ R | R \subseteq Q \}$

$E(R) = \{ q | q$ can be reached from $R$ by traveling along 0 or more $\varepsilon$ arrows$\}.$

$δ'(R,a) = \{ q \in Q | \exists r \in R, q \in E(δ(r,a)) \}, \forall a \neq \varepsilon$

$q'_0 = E(q_0)$
NFA-DFA equivalence
(with empty transitions)

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA accepting language $A$. We construct a DFA $M = (Q', \Sigma, \delta', q'_0, F')$ accepting $A$ as well.

- $Q' = \mathcal{P}(Q) = \{ R | R \subseteq Q \}$

$E(R) = \{ q | q$ can be reached from $R$ by traveling along $0$ or more $\varepsilon$ arrows\}.

- $\delta'(R, a) = \{ q \in Q | \exists r \in R, q \in E(\delta(r, a)) \}, \forall a \neq \varepsilon$

- $q'_0 = E(q_0)$

- $F' = \{ R \in Q' | R \cap F \neq \emptyset \}$
\( N_1 \)

\[ q_1 \xrightarrow{1} q_2 \xrightarrow{0, \varepsilon} q_3 \xrightarrow{1} q_4 \]

\( q \{ 1, 2, 3, 4 \} \)
The diagram represents a non-deterministic finite automaton (NFA) labeled $N_1$. The states are $q_1$, $q_2$, $q_3$, and $q_4$, with transitions as follows:

- From $q_1$ to $q_2$ with input $1$.
- From $q_2$ to $q_1$ and $q_3$ with input $0, \epsilon$.
- From $q_3$ to $q_4$ with input $1$.
- From $q_4$ to itself with input $0, 1$.

The alphabet consists of $0, 1, \epsilon$. The initial state is $q_1$, and the accepting states are $q_4$. Other states are $q_2$ and $q_3$.
\[ N_1 \]

- \( q_1 \) connected to \( q_2 \) with edge labeled 1.
- \( q_2 \) connected to \( q_3 \) with edge labeled \( 0, \varepsilon \).
- \( q_3 \) connected to \( q_4 \) with edge labeled 1.
- \( q_4 \) has a self-loop labeled \( 0,1 \).

States:
- \( q_\emptyset \)
- \( q_{\{1\}} \)
- \( q_{\{2\}} \)
- \( q_{\{3\}} \)
- \( q_{\{4\}} \)
- \( q_{\{1,2,3,4\}} \)
The automaton $N_1$ is defined as follows:

- The initial state is $q_1$.
- The states are:
  - $q_0$
  - $q_{\{1\}}$
  - $q_{\{1,2\}}$
  - $q_{\{1,3\}}$
  - $q_{\{1,4\}}$
  - $q_{\{2\}}$
  - $q_{\{3\}}$
  - $q_{\{4\}}$
  - $q_{\{1,2,3\}}$
  - $q_{\{1,2,3,4\}}$
- The transitions are:
  - $q_1 \rightarrow q_2$ on input $1$.
  - $q_2 \rightarrow q_3$ on input $0, \varepsilon$.
  - $q_3 \rightarrow q_4$ on input $1$.
  - The loop from $q_4$ on input $0,1$.

The automaton is defined over the alphabet $\{0,1\}$.
The image depicts a formal language automaton labeled $N_1$. The automaton consists of states $q_1, q_2, q_3, q_4$, with transitions labeled by symbols $0, 1, \varepsilon$. The automaton transitions according to the input string $010110$. The states are labeled with subsets of the input alphabet, indicating the set of states the automaton is in following a specific input sequence.
Regular Operations:
Kleene's theorem (NFA)
Regular Operations:
Kleene's theorem
Regular Operations: Kleene’s theorem

**Theorem 1.45**

The class of regular languages is closed under the union operation.
THEOREM 1.45

The class of regular languages is closed under the union operation.
Kleene’s theorem
Let \( N_A=(Q_A, \Sigma, \delta_A, q_{0A}, F_A) \) be a NFA accepting \( L_A \) and \( N_B=(Q_B, \Sigma, \delta_B, q_{0B}, F_B) \) be a NFA accepting \( L_B \) \( (Q_A \cap Q_B = \emptyset) \).
Let $N_A = (Q_A, \Sigma, \delta_A, q_{0A}, F_A)$ be a NFA accepting $L_A$ and
\[ N_B = (Q_B, \Sigma, \delta_B, q_{0B}, F_B) \]
be a NFA accepting $L_B$ ($Q_A \cap Q_B = \emptyset$).

Consider $N_U = (\{q_0\} \cup Q_A \cup Q_B, \Sigma, \delta_U, q_0, F_U)$ where
\[ \delta_U(q_0, \varepsilon) = \{q_{0A}, q_{0B}\}, \delta_U(q_0, a) = \emptyset \text{ for all } a \neq \varepsilon, \]
\[ \delta_U(q, a) = \delta_X(q, a) \text{ for all } q \in Q_X, X \in \{A, B\}, \text{ and all } a, \]
\[ F_U = F_A \cup F_B. \]
Let \( N_A = (Q_A, \Sigma, \delta_A, q_{0A}, F_A) \) be a NFA accepting \( L_A \) and \( N_B = (Q_B, \Sigma, \delta_B, q_{0B}, F_B) \) be a NFA accepting \( L_B \) \( (Q_A \cap Q_B = \emptyset) \).

Consider \( N_U = (\{q_0\} \cup Q_A \cup Q_B, \Sigma, \delta_U, q_0, F_U) \) where

\[
\delta_U(q_0, \varepsilon) = \{q_{0A}, q_{0B}\}, \quad \delta_U(q_0, a) = \emptyset \text{ for all } a \neq \varepsilon,
\]

\[
\delta_U(q, a) = \delta_X(q, a) \text{ for all } q \in Q_X, X \in \{A, B\}, \text{ and all } a,
\]

\( F_U = F_A \cup F_B \).

\( L_U = L_A \cup L_B \).
Example

$N_1$

$N_2$
Example

$N_1$

$q_{1A}$ \[ \xrightarrow{0,1} \] \[ \xrightarrow{1} \] $q_{2A}$ \[ \xrightarrow{0,\varepsilon} \] $q_{3A}$ \[ \xrightarrow{1} \] $q_{4A}$

$N_2$

$q_{1B}$ \[ \xrightarrow{0,1} \] \[ \xrightarrow{1} \] $q_{2B}$ \[ \xrightarrow{0,1} \] $q_{3B}$ \[ \xrightarrow{0,1} \] $q_{4B}$
Example

\[ N_1 \]

\[ q_{1A} \rightarrow 1 \rightarrow q_{2A} \rightarrow 0, \varepsilon \rightarrow q_{3A} \rightarrow 1 \rightarrow q_{4A} \]

\[ N_2 \]

\[ q_{1B} \rightarrow 1 \rightarrow q_{2B} \rightarrow 0, 1 \rightarrow q_{3B} \rightarrow 0, 1 \rightarrow q_{4B} \]
Example
Example

\[ N_1 \]

\[ q_0 \]

\[ q_{1A} \rightarrow_{0,1} q_{2A} \rightarrow_{1} q_{3A} \rightarrow_{0,\varepsilon} q_{4A} \]

\[ N_2 \]

\[ q_{1B} \rightarrow_{0,1} q_{2B} \rightarrow_{1} q_{3B} \rightarrow_{0,1} q_{4B} \]
Example
Regular Operations:
Kleene's theorem
Regular Operations: Kleene's theorem

**THEOREM 1.47**

The class of regular languages is closed under the concatenation operation.
THEOREM 1.47

The class of regular languages is closed under the concatenation operation.
Kleene's theorem
Kleene's theorem

Let $N_A = (Q_A, \Sigma, \delta_A, q_0A, F_A)$ be a NFA accepting $L_A$ and $N_B = (Q_B, \Sigma, \delta_B, q_0B, F_B)$ be a NFA accepting $L_B$ ($Q_A \cap Q_B = \emptyset$).
Let \( N_A = (Q_A, \Sigma, \delta_A, q_{0A}, F_A) \) be a NFA accepting \( L_A \) and \( N_B = (Q_B, \Sigma, \delta_B, q_{0B}, F_B) \) be a NFA accepting \( L_B \) (\( Q_A \cap Q_B = \emptyset \)).

Consider \( N_C = (Q_A \cup Q_B, \Sigma, \delta_C, q_{0A}, F_B) \) where

\[
\begin{align*}
\delta_C(q,a) &= \delta_B(q,a) \text{ for all } q \in Q_B, \text{ all } a, \\
\delta_C(q,a) &= \delta_A(q,a) \text{ for all } q \in Q_A, \text{ all } a \neq \varepsilon, \\
\delta_C(q,\varepsilon) &= \delta_A(q,\varepsilon) \text{ for all } q \in Q_A \setminus F_A, \\
\delta_C(q,\varepsilon) &= \delta_A(q,\varepsilon) \cup \{q_{0B}\} \text{ for all } q \in F_A.
\end{align*}
\]
Let \( N_A = (Q_A, \Sigma, \delta_A, q_0A, F_A) \) be a NFA accepting \( L_A \) and \( N_B = (Q_B, \Sigma, \delta_B, q_0B, F_B) \) be a NFA accepting \( L_B \) (\( Q_A \cap Q_B = \emptyset \)).

Consider \( N_C = (Q_A \cup Q_B, \Sigma, \delta_C, q_0A, F_B) \) where

\[
\begin{align*}
\delta_C(q, a) & = \delta_B(q, a) \text{ for all } q \in Q_B, \text{ all } a, \\
\delta_C(q, a) & = \delta_A(q, a) \text{ for all } q \in Q_A, \text{ all } a \neq \varepsilon, \\
\delta_C(q, \varepsilon) & = \delta_A(q, \varepsilon) \text{ for all } q \in Q_A \setminus F_A, \\
\delta_C(q, \varepsilon) & = \delta_A(q, \varepsilon) \cup \{q_0B\} \text{ for all } q \in F_A.
\end{align*}
\]

\( L_C = L_A \circ L_B. \)
Example

\[ N_1 \]

\[ q_{1A} \xrightarrow{1} q_{2A} \xrightarrow{0, \varepsilon} q_{3A} \xrightarrow{1} q_{4A} \]

\[ N_2 \]

\[ q_{1B} \xrightarrow{1} q_{2B} \xrightarrow{0, 1} q_{3B} \xrightarrow{0, 1} q_{4B} \]
Example

\[ N_1 \]
\[ q_{1A} \quad 1 \quad q_{2A} \quad 0, ε \quad q_{3A} \quad 1 \quad q_{4A} \]

\[ N_2 \]
\[ q_{1B} \quad 1 \quad q_{2B} \quad 0, 1 \quad q_{3B} \quad 0, 1 \quad q_{4B} \]
Example

\( N_1 \)

\[ q_{1A} \xrightarrow{0,1} q_{2A} \xrightarrow{1} q_{3A} \xrightarrow{0,\varepsilon} q_{4A} \]

\( N_2 \)

\[ q_{1B} \xrightarrow{0,1} q_{2B} \xrightarrow{1} q_{3B} \xrightarrow{0,1} q_{4B} \]
Example
Example
Regular Operations: Kleene's theorem

**Theorem 1.49**

The class of regular languages is closed under the star operation.
THEOREM 1.49

The class of regular languages is closed under the star operation.
Kleene’s theorem
Kleene's theorem

Let $N_A = (Q_A, \Sigma, \delta_A, q_{0A}, F_A)$ be a NFA accepting $L_A$. 
Kleene's theorem

Let $N_A = (Q_A, \Sigma, \delta_A, q_{0A}, F_A)$ be a NFA accepting $L_A$.

Consider $N_S = (Q_A \cup \{q_0\}, \Sigma, \delta_S, q_0, F_A \cup \{q_0\})$ where

- $\delta_S(q_0, \varepsilon) = q_{0A}$, and $\delta_S(q_0, a) = \emptyset$ for all $a \neq \varepsilon$,
- $\delta_S(q, a) = \delta_A(q, a)$ for all $q \in Q_A \setminus F_A$, all $a$,
- $\delta_S(q, \varepsilon) = \delta_A(q, \varepsilon) \cup \{q_{0A}\}$ for all $q \in F_A$,
- $\delta_S(q, a) = \delta_A(q, a)$ for all $q \in F_A$, all $a \neq \varepsilon$. 
Let $N_A = (Q_A, \Sigma, \delta_A, q_{0A}, F_A)$ be a NFA accepting $L_A$.

Consider $N_S = (Q_A \cup \{q_0\}, \Sigma, \delta_S, q_0, F_A \cup \{q_0\})$ where

- $\delta_S(q_0, \varepsilon) = q_{0A}$, and $\delta_S(q_0, a) = \emptyset$ for all $a \neq \varepsilon$,
- $\delta_S(q, a) = \delta_A(q, a)$ for all $q \in Q_A \setminus F_A$, all $a$,
- $\delta_S(q, \varepsilon) = \delta_A(q, \varepsilon) \cup \{q_{0A}\}$ for all $q \in F_A$,
- $\delta_S(q, a) = \delta_A(q, a)$ for all $q \in F_A$, all $a \neq \varepsilon$.

$L_S = (L_A)^*.$
Example

\[N_1\]

\[q_1 \xrightarrow{1} q_2 \xrightarrow{0,\varepsilon} q_3 \xrightarrow{1} q_4\]
Example
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