4.4 Let $A_{\varepsilon_{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG that generates } \varepsilon \}$. Show that $A_{\varepsilon_{CFG}}$ is decidable.

4.13 Let $A = \{\langle R, S \rangle \mid R \text{ and } S \text{ are regular expressions and } L(R) \subseteq L(S)\}$. Show that $A$ is decidable.

4.28 Let $C = \{\langle G, x \rangle \mid G \text{ is a CFG } x \text{ is a substring of some } y \in L(G)\}$. Show that $C$ is decidable. (Hint: An elegant solution to this problem uses the decider for $E_{CFG}$.)

5.16 Let $\Gamma = \{0, 1, \omega\}$ be the tape alphabet for all TMs in this problem. Define the busy beaver function $BB : \mathbb{N} \rightarrow \mathbb{N}$ as follows. For each value of $k$, consider all $k$-state TMs that halt when started with a blank tape. Let $BB(k)$ be the maximum number of 1s that remain on the tape among all of these machines. Show that $BB$ is not a computable function.

+ Prove that $BB(k)$ grows faster than any computable function $f(k)$, i.e. $\lim_{k \to \infty} f(k)/BB(k)=0$.

5.19 In the silly Post Correspondence Problem, $SPCP$, in each pair the top string has the same length as the bottom string. Show that the $SPCP$ is decidable.

5.20 Prove that there exists an undecidable subset of $\{1\}^*$.

5.21 Let $AMBIG_{CFG} = \{\langle G \rangle \mid G \text{ is an ambiguous CFG}\}$. Show that $AMBIG_{CFG}$ is undecidable. (Hint: Use a reduction from $PCP$. Given an instance

$$P = \left\{ \begin{bmatrix} t_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} t_2 \\ b_2 \end{bmatrix}, \ldots, \begin{bmatrix} t_k \\ b_k \end{bmatrix} \right\},$$

of the Post Correspondence Problem, construct a CFG $G$ with the rules

$$S \rightarrow T \mid B$$
$$T \rightarrow t_1 Ta_1 \mid \cdots \mid t_k Ta_k \mid a_1 \mid \cdots \mid t_k a_k$$
$$B \rightarrow b_1 Ba_1 \mid \cdots \mid b_k Ba_k \mid a_1 \mid \cdots \mid b_k a_k$$

where $a_1, \ldots, a_k$ are new terminal symbols. Prove that this reduction works.)
5.31 Let

\[ f(x) = \begin{cases} 
3x + 1 & \text{for odd } x \\
\frac{x}{2} & \text{for even } x 
\end{cases} \]

for any natural number \( x \). If you start with an integer \( x \) and iterate \( f \), you obtain a sequence, \( x, f(x), f(f(x)), \ldots \). Stop if you ever hit 1. For example, if \( x = 17 \), you get the sequence 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Extensive computer tests have shown that every starting point between 1 and a large positive integer gives a sequence that ends in 1. But, the question of whether all positive starting points end up at 1 is unsolved; it is called the 3x + 1 problem.

Suppose that \( A_{TM} \) were decidable by a TM \( H \). Use \( H \) to describe a TM that is guaranteed to state the answer to the 3x + 1 problem.

6.19 Recall the Post correspondence problem that we defined in Section 5.2 and its associated language \( PCP \). Show that \( PCP \) is decidable relative to \( A_{TM} \).

7.21 Let \( DOUBLE-SAT = \{\langle \phi \rangle | \phi \text{ has at least two satisfying assignments} \} \). Show that \( DOUBLE-SAT \) is NP-complete.

7.27 A coloring of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color. Let

\[ 3COLOR = \{\langle G \rangle | \text{the nodes of } G \text{ can be colored with three colors such that no two nodes joined by an edge have the same color} \} \]

Show that \( 3COLOR \) is NP-complete. (Hint: Use the following three subgraphs.)