## COMP 330A 2019, Assignment 4 Due Thursday, November 28th 2019 23:59

\_\_\_\_\_

- [8%]
- **4.4** Let  $A\varepsilon_{\mathsf{CFG}} = \{\langle G \rangle | G \text{ is a CFG that generates } \varepsilon \}$ . Show that  $A\varepsilon_{\mathsf{CFG}}$  is decidable.
- **[10%**]
- **4.13** Let  $A = \{\langle R, S \rangle | R \text{ and } S \text{ are regular expressions and } L(R) \subseteq L(S) \}$ . Show that A is decidable.
- [10%]
- **4.28** Let  $C = \{\langle G, x \rangle | G \text{ is a CFG } x \text{ is a substring of some } y \in L(G) \}$ . Show that C is decidable. (Hint: An elegant solution to this problem uses the decider for  $E_{\mathsf{CFG}}$ .)
- [8%]
- 5.16 Let  $\Gamma = \{0, 1, \sqcup\}$  be the tape alphabet for all TMs in this problem. Define the *busy* beaver function  $BB: \mathcal{N} \longrightarrow \mathcal{N}$  as follows. For each value of k, consider all k-state TMs that halt when started with a blank tape. Let BB(k) be the maximum number of 1s that remain on the tape among all of these machines. Show that BB is not a computable function.
- [+5%]
- + Prove that BB(k) grows faster than any computable function f(k), i.e.  $\lim_{k\to\infty} f(k)/BB(k)=0$ .
- **[10%**]
- **5.19** In the *silly Post Correspondence Problem*, *SPCP*, in each pair the top string has the same length as the bottom string. Show that the *SPCP* is decidable.
- [10%]
- **5.20** Prove that there exists an undecidable subset of  $\{1\}^*$ .
- [10%]
- **5.21** Let  $AMBIG_{CFG} = \{\langle G \rangle | G \text{ is an ambiguous CFG} \}$ . Show that  $AMBIG_{CFG}$  is undecidable. (Hint: Use a reduction from PCP. Given an instance

$$P = \left\{ \left[ \frac{t_1}{b_1} \right], \left[ \frac{t_2}{b_2} \right], \dots, \left[ \frac{t_k}{b_k} \right] \right\},\,$$

of the Post Correspondence Problem, construct a CFG G with the rules

$$S \to T \mid B$$

$$T \to t_1 T \mathbf{a}_1 \mid \cdots \mid t_k T \mathbf{a}_k \mid t_1 \mathbf{a}_1 \mid \cdots \mid t_k \mathbf{a}_k$$

$$B \to b_1 B \mathbf{a}_1 \mid \cdots \mid b_k B \mathbf{a}_k \mid b_1 \mathbf{a}_1 \mid \cdots \mid b_k \mathbf{a}_k$$

where  $a_1, \ldots, a_k$  are new terminal symbols. Prove that this reduction works.)

\_\_\_\_\_\_

$$f(x) = \begin{cases} 3x + 1 & \text{for odd } x \\ x/2 & \text{for even } x \end{cases}$$

for any natural number x. If you start with an integer x and iterate f, you obtain a sequence,  $x, f(x), f(f(x)), \ldots$  Stop if you ever hit 1. For example, if x = 17, you get the sequence 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Extensive computer tests have shown that every starting point between 1 and a large positive integer gives a sequence that ends in 1. But, the question of whether all positive starting points end up at 1 is unsolved; it is called the 3x + 1 problem.

Suppose that  $A_{\mathsf{TM}}$  were decidable by a TM H. Use H to describe a TM that is guaranteed to state the answer to the 3x+1 problem.

[8%]

**6.19** Recall the Post correspondence problem that we defined in Section 5.2 and its associated language PCP. Show that PCP is decidable relative to  $A_{TM}$ .

------

[8%]

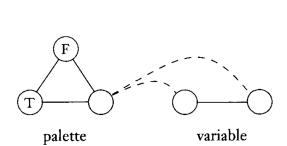
**7.21** Let DOUBLE- $SAT = \{\langle \phi \rangle | \phi \text{ has at least two satisfying assignments} \}$ . Show that DOUBLE-SAT is NP-complete.

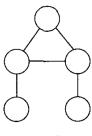
[10%]

**7.27** A *coloring* of a graph is an assignment of colors to its nodes so that no two adjacent nodes are assigned the same color. Let

 $3COLOR = \{\langle G \rangle | \text{ the nodes of } G \text{ can be colored with three colors such that no two nodes joined by an edge have the same color}.$ 

Show that *3COLOR* is NP-complete. (Hint: Use the following three subgraphs.)





OR-gadget