Solution set for Assignment 1

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Exercise 1

Consider the following algorithm:

While some hospitals h_i has available positions (and didn't offer a position to all the students) **do**
    h_i offers a position to the next student s on its preference list
    If s is free **then**
        s accepts the position and the number of available positions of h_i decreases by 1.
    else (s is already committed with another hospital h_j)
        If s prefers h_i to h_j **then**
            s accepts h_i's offer and becomes committed to h_i
            the number of available positions of h_i decreases by 1
            the number of available positions of h_j increases by 1
        else (s prefers h_j to h_i)
            s turns down h_i's offer.

The algorithm always returns a correct assignment of students to hospitals. In fact, suppose that each hospital as p_i's available positions and suppose that there are in total n students. Then by assumption we know that $\sum_{i=1}^{m} p_i < n$. Now, suppose that the algorithm terminates with an assignment where not all the available solutions are filled. Then it must be that all the hospitals that did not fill some position have offered one to every students. Thus, it must be that $n > \sum_{i=1}^{m} p_i$, which contradicts our hypothesis.
Now we need to prove that the algorithm returns a stable assignment. Again we will prove it by contradiction; suppose that the assignment is not stable, but it has an instability of the first type. That is, suppose:

1. s is assigned to h
2. s' is free
3. h prefers s' to s

Then, since h prefers s' to s, h would have offered a position to s' before offering it to s. Thus, since s got a position, we can conclude that also s' must be assigned to some hospital reaching a contradiction.
Finally, suppose that there is an instability of the second type, that is:

1. s is assigned to h
2. s' is assigned to h'
3. h prefers s' to s
4. s' prefers h to h'

From (3), we know that either h offered a position to s', or h hired all residents that it prefers to s'. Then, using fact (1), we can exclude the second possibility and conclude that h must have offered a position to s'. Since s' is not assigned to h, he must have rejected h's offer in favor of another hospital h'' which he prefers to h. Finally, given (2), we can conclude that s' prefers h' to all the previous hospitals that offered him a position, that is, we can conclude that s' prefers h' to h reaching a contradiction.

Exercise 2