2. Either prove the following statement or exhibit a counter-example.

The solutions produced by both algorithms are equal if and only if this is the only solution to the input instance.

Two things have to be proved here:

A) (this is the easier part)

If there is only one solution to the input instance
then
the solutions produced by both algorithms are equal.

B) (this is the harder part)

If the solutions produced by both algorithms are equal
then
this is the only solution to the input instance.

And remember that by the contrapositive formulation the latter one is equivalent to

If there are more than one solution to the input instance
then
the solutions produced by both algorithms must be distinct.

A) Since we have already proved in class that both algorithms output a solution, if there is only one solution then they must both output that unique one.

B) If there is more than one solution then let M₀ and M₁ be two distinct stable matchings for the given input preference-lists and let M₀ be the solution that is men-optimal. We will show that M₀ cannot be women-optimal at the same time.

Take the men in some arbitrary order and select the first man m who is matched with a different woman in each of M₀ and M₁. Let w₀ and w₁ be m’s matches in M₀ and M₁. By the men-optimality of M₀ we conclude that w₀ is m’s preferred valid partner. Let m’ be w₀’s partner in M₁. Since M₁ is stable, it must be that w₀ prefers m’ to m (otherwise (w₀-m’) would be unstable in M₁). Now we know that both m and m’ are valid partners of w₀ but that w₀ prefers m’ to m. In consequence M₀ cannot be women-optimal as well because w₀ is not matched with her preferred valid partner (m is not her favorite valid partner).

QED