COMP 251 2016, Assignment 1

2. Either prove the following statement or exhibit a counter-example.

The solutions produced by both algorithms are equal
if and only if
this is the only solution to the input instance.

Two things have to be proved here:

A) (this is the easier part)
If there is only one solution to the input instance
then
the solutions produced by both algorithms are equal.

B) (this is the harder part)
If the solutions produced by both algorithms are equal
then
this is the only solution to the input instance.

And remember that by the contrapositive formulation the latter one is equivalent to

If there are more than one solution to the input instance
then
the solutions produced by both algorithms must be distinct.

A) Since we have already proved in class that both algorithms output a solution, if there is only one solution then they must both output that unique one.

B) If there is more than one solution then let M_0 and M_1 be two distinct stable matchings for the given input preference-lists and let M_0 be the solution that is men-optimal. We will show that M_0 cannot be women-optimal at the same time.

Take the men in some arbitrary order and select the first man m who is matched with a different woman in each of M_0 and M_1. Let w_0 and w_1 be m's matches in M_0 and M_1. By the men-optimality of M_0 we conclude that w_0 is m's preferred valid partner. Let m' be w_0's partner in M_1. Since M_1 is stable, it must be that w_0 prefers m' to m (otherwise (w_0-m') would be unstable in M_1). Now we know that both m and m' are valid partners of w_0 but that w_0 prefers m' to m. In consequence M_0 cannot be women-optimal as well because w_0 is not matched with her preferred valid partner (m is not her favorite valid partner). **QED**