Faculty of Science
Final Examination

Computer Science 308-251B
Data Structures and Algorithms

Examiner:	Prof. Claude Crépeau	**Date:**	April 20, 2005
Associate Examiner:	Lecturer Martin Courchesne	**Time:**	14:00 – 17:00

INSTRUCTIONS:

This examination is worth 50% of your final grade.
The total of all questions is 105 points.
Each question is assigned a value found in brackets next to it.
O P E N • B O O K S •/• O P E N • N O T E S
Faculty standard calculator permitted only.
This examination consists of 4 pages including title page.
This examination consists of 7 questions.

SUGGESTION : read all the questions and their values before you start.

[15%]

1) As promised !

Let

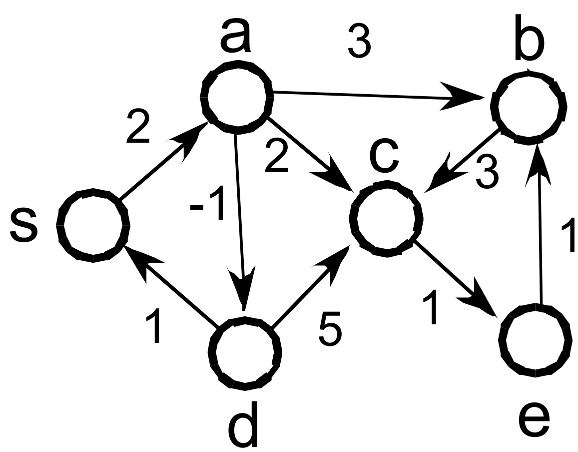
$$T(n) = \begin{cases} 1 & \text{if } n=1,2 \\ T(\lceil n/4 \rceil) + T(\lceil 2n/3 \rceil) + \Theta(n) & \text{if } n>2 \end{cases}$$

Prove using constructive induction that T(n) is $\Theta(n)$.

[15%]

2)

Consider the following graph.



Find the shortest path from source s to every vertex $v \in \{s,a,b,c,d,e\}$ in the graph. Choose the algorithm you wish to use, and show the step by step progression of the algorithm on this instance.

[15%]

3)

Consider an infinite family of graphs $\{G_i=(V_i,E_i),W_i\}_{i>0}$ such that $|V_i|=i$, and W_i is the weight function of the edges in E_i. Assume the weight of any edge (u,v) in E_i is such that $0 < w_i(u,v) < |E_i|$. Explain how to modify Kruskal's algorithm so that it is asymptotically a better choice than (unmodified) Prim's algorithm. Compare the running times.

Hint: first understand why $w_i(u,v) < |E_i|$ makes a difference; what is more efficient then ?

[15%]

4)

Consider ways of computing the values of the binomial coefficients $\binom{n}{k}$. First remember that $\binom{n}{k} = n! / k!\,(n-k)!$, and a basic property of binomial coefficients $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

A) Show that if we blindly use this recursive formula and the fact that $\binom{n}{1}=n$, it will take exponential time in n to compute $\binom{n}{k}$ for certain values of k.

B) Give a dynamic programming algorithm to compute $\binom{n}{k}$ and analyze its running time.

[15%]

5)
Consider the following table of frequencies of Café's in Montréal:

Café	**Acronym**	**Frequency**
Brûlerie St-Denis	BD	14%
Café Dépôt	CD	4%
Presse-café	PC	9%
Second Cup	SC	22%
Starbuck Cofee	SC	2%
Tim Hortons	TH	20%
Van Houtte	VH	13%
All Others	AO	16%

Build a Huffman tree from these frequencies and use these acronyms as labels of vertices. Show the step by step progression of the HUFFMAN algorithm on this instance.

[10pts]

6)
For each statement, say if it is *true* or *false*.

Correct = +1 pt, Incorrect = -0.5 pt, No answer = 0 pt, Minimum Total= 0 pt.

(a) Red-Black trees are important to get better sorting algorithms.

(b) Finding large prime numbers is mathematically beautiful but practically useless.

(c) The Longest Common Subsequence of X and Y is unique.

(d) The "disjoint sets" data structures of chapter 21 are fundamental to Prim's algorithm for Minimum Spanning Trees.

(e) We can easily adapt Strassen's sub-cubic algorithm for matrix multiplication to improve All-pairs Shortest Paths algorithms.

(f) In an array of unsorted integers we can find the element of rank $\lceil \sqrt{n} \rceil$ in worst case time O(n).

(g) The worst case to the basic "HIRE-ASSISTANT" algorithm of Chapter 5 is when the candidates are presented in increasing order of quality.

(h) In Radix sort it is mandatory that the underlying sorting algorithm be *stable*.

(i) The family of hash functions defined as $Ax \oplus b$ on n-bit vectors x, where A is an nxn invertible binary matrix, and b an arbitrary n-bit vector, is universal.

(j) A tree is a forest.

[20%] **7)**
Consider a sorting algorithm A such that the expected running time of A is $< a \bullet n \log n$ to sort any array of n elements. Suppose another sorting algorithm C is such that the worst case running time of C is $< c \bullet n \log n$ to sort any array of n elements, but c is much larger than a.

A) Show that the probability that algorithm A will take more than $c \bullet n \log n$ time is at most a/c.

Consider the algorithm B that runs A upto $5c \bullet n \log n$ steps and then runs C instead.

B) Show that the worst case running time of B is no more than $6c \bullet n \log n$.

C) Show that the expected running time of B is no more than $(1.2) \bullet a \bullet n \log n$.

D) Conclude from this example that for any $\varepsilon > 0$, we can design from A and C another algorithm D such that the worst case running time of D is $O(n \log n)$, while its expected running time is no more than $(1+\varepsilon) \bullet a \bullet n \log n$.