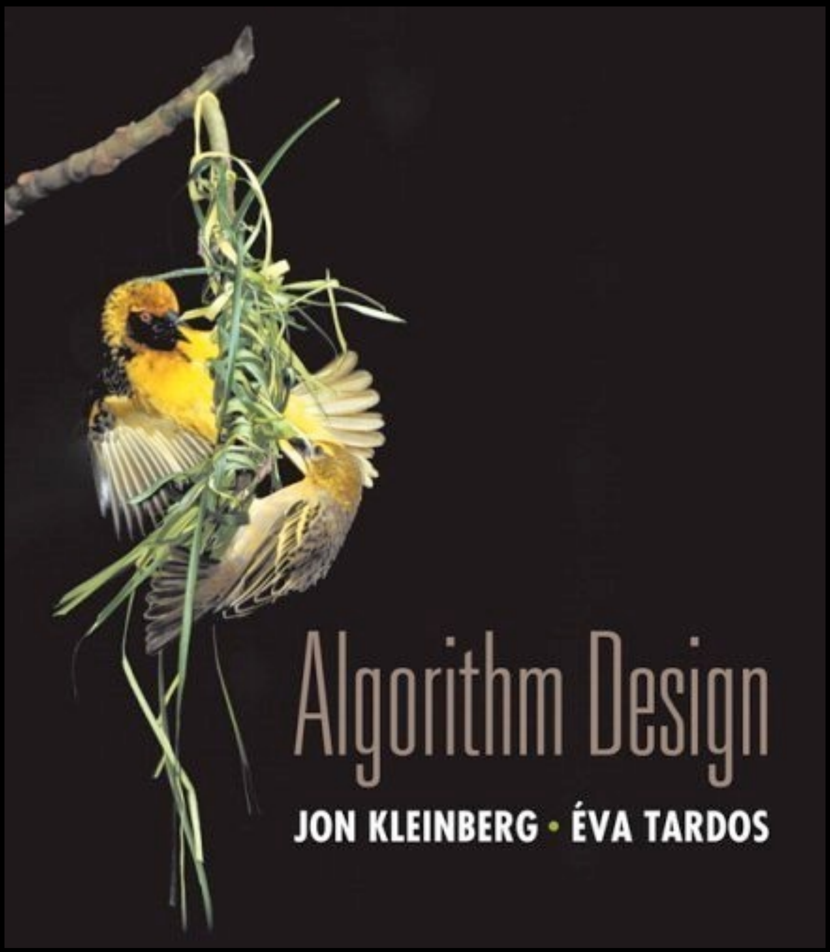
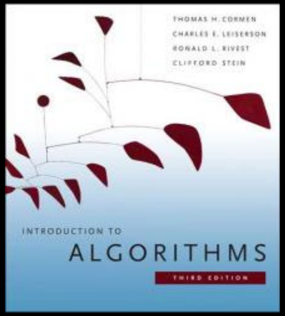


Chapter 3

Graphs

CLRS 12-13



PEARSON Addison Wesley

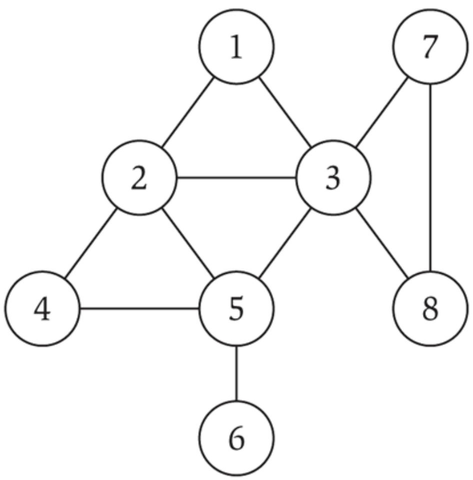
Slides by Kevin Wayne.

3.1 Basic Definitions and Applications

Undirected Graphs

Undirected graph. G = (V, E)

- V = nodes.
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: $n = |V|$, $m = |E|$.



V = { 1, 2, 3, 4, 5, 6, 7, 8 }

E = { (1,2), (1,3), (2,3), (2,4), (2,5),
(3,5), (3,7), (3,8), (4,5), (5,6), (7,8) }

n = 8

m = 11

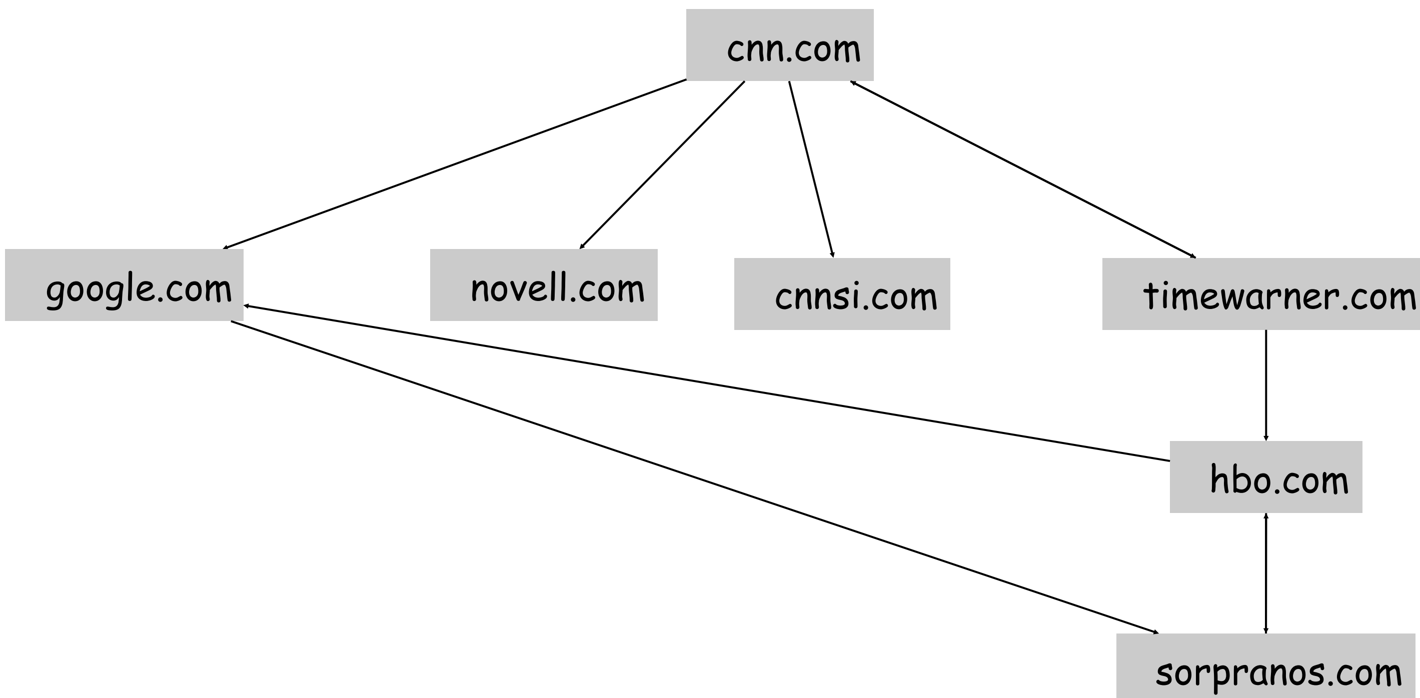
Some Graph Applications

Graph	Nodes	Edges
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

World Wide Web

Web graph.

- Node: web page.
- Edge: hyperlink from one page to another.



9-11 Terrorist Network

Social network graph.

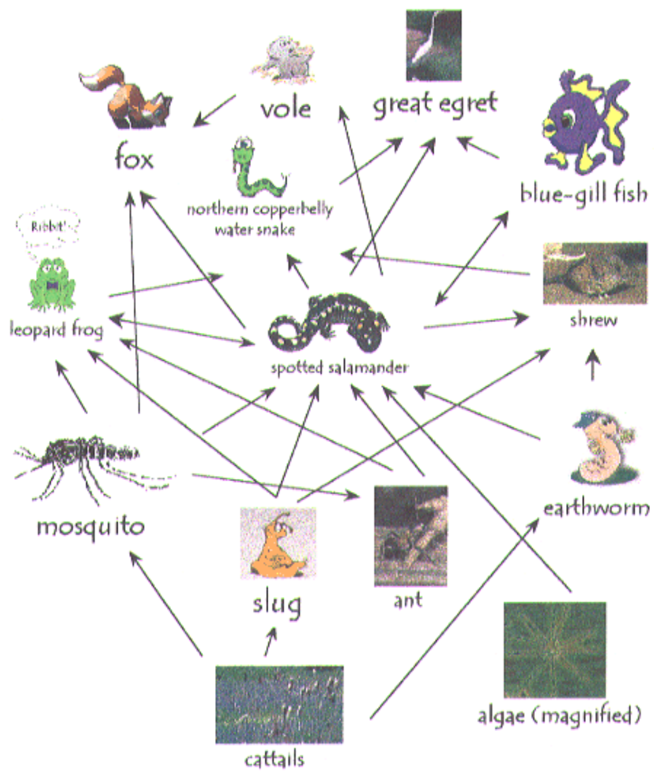
- Node: people.
- Edge: relationship between two people.

Reference: Valdis Krebs, http://www.firstmonday.org/issues/issue7_4/krebs

Ecological Food Web

Food web graph.

- Node = species.
- Edge = from prey to predator.



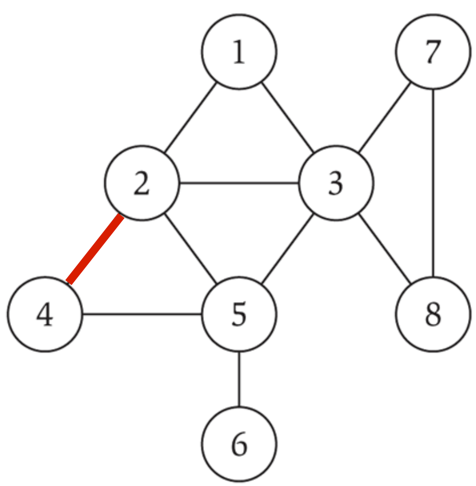
Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff

Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with A_{uv} = 1 iff (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
  (Checking k pairs (u,v) will cost $\Theta(k)$ time.)
- Identifying all edges takes $\Theta(n^2)$ time.

n = number of vertices



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency List

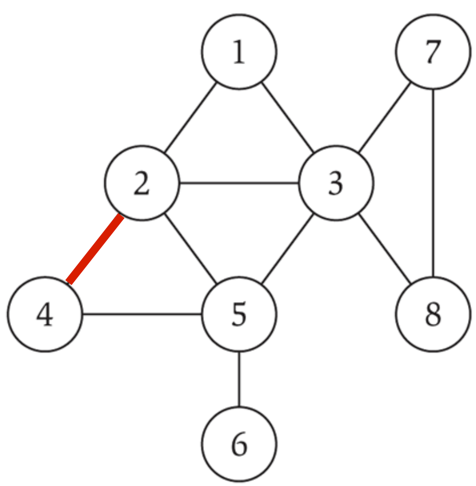
Adjacency list. Node indexed array of lists.

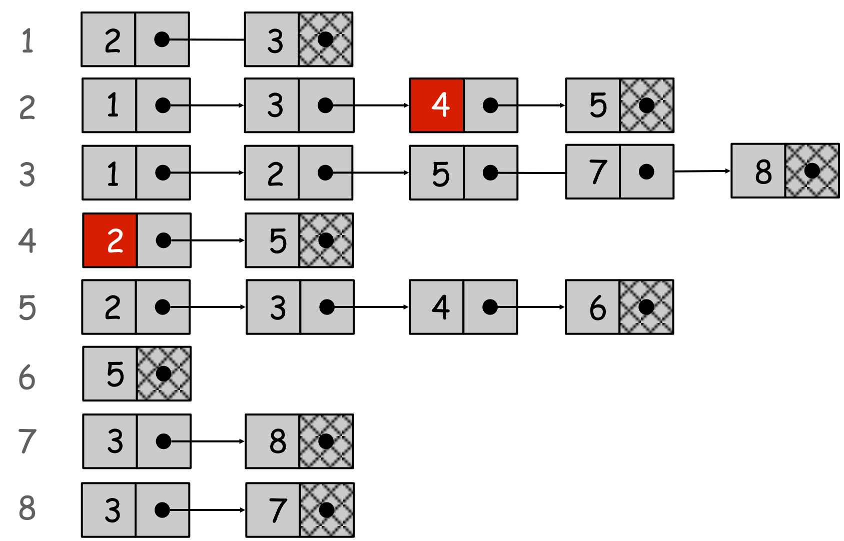
- Two representations of each edge.
- Space proportional to m + n.

degree = number of neighbours of u

- Checking if (u, v) is an edge takes O(deg(u)) time.
  <u>(Checking k pairs (u,v) may cost up to Θ(kn) time.)</u>
- Identifying all edges takes Θ(m + n) time.

n = number of vertices



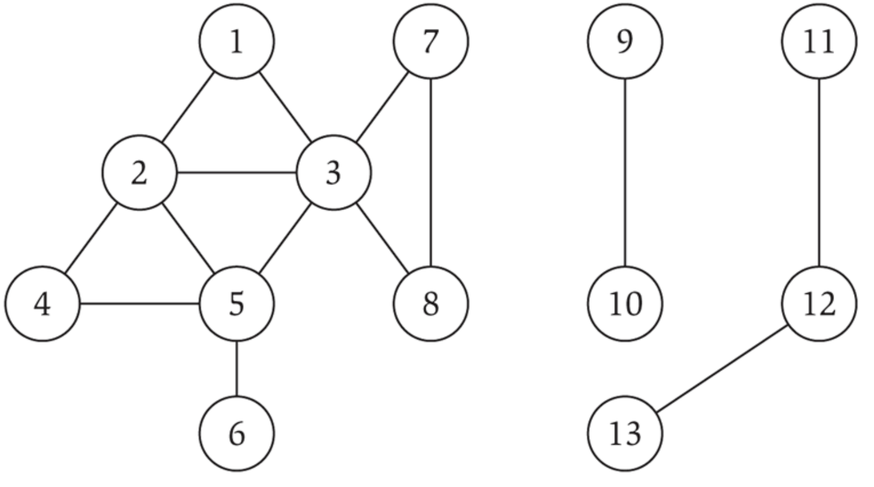


Paths and Connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.

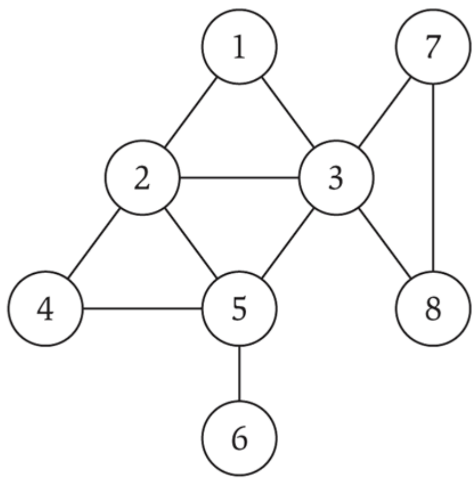
Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first k-1 nodes are all distinct.



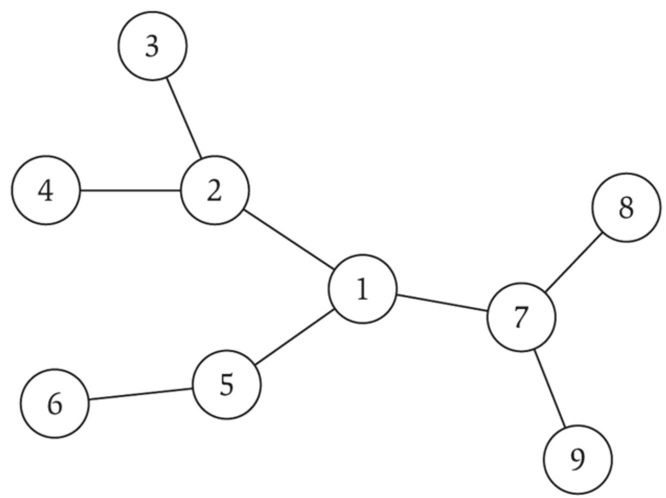
cycle C = 1-2-4-5-3-1

Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

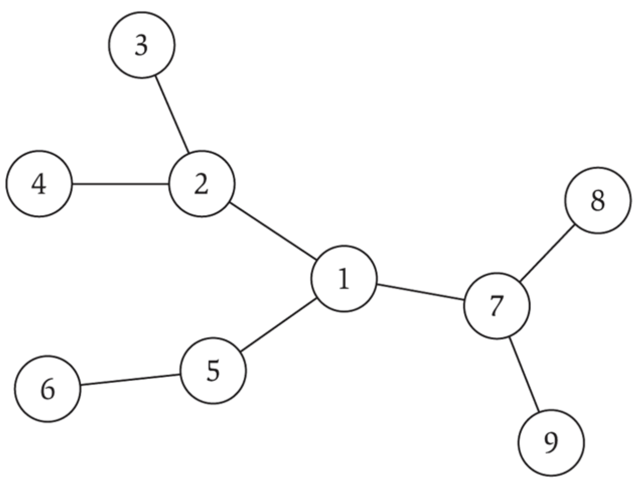
- G is connected.
- G does not contain a cycle.
- G has n-1 edges.



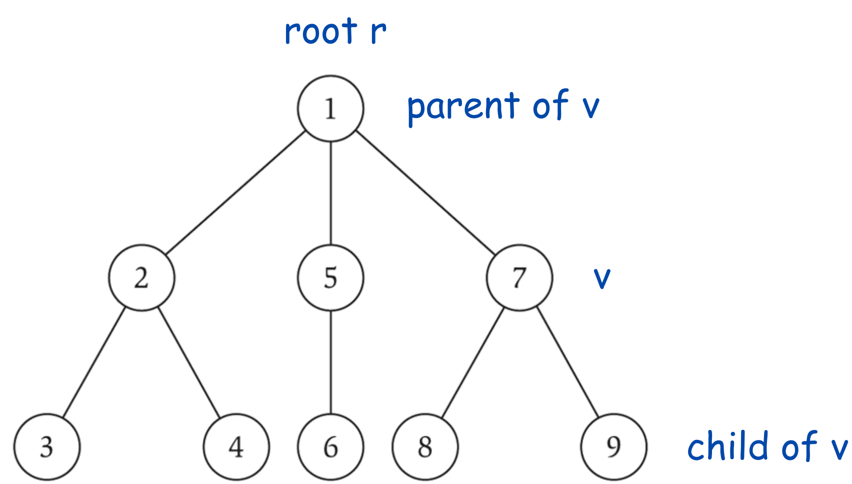
Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.

Importance. Models hierarchical structure.



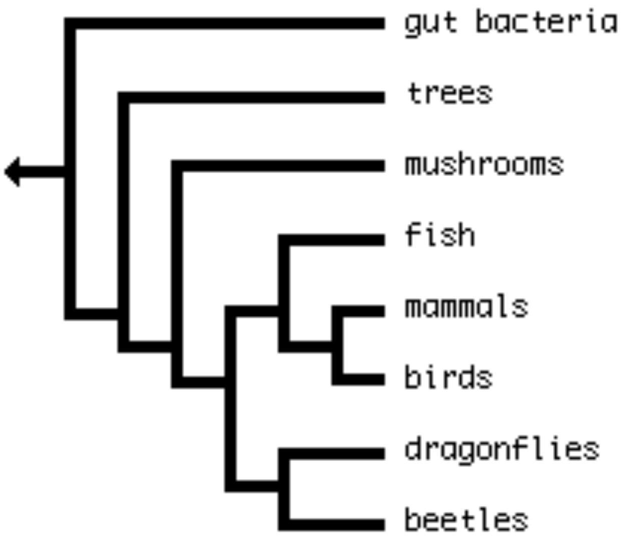
a tree



the same tree, rooted at 1

Phylogeny Trees

Phylogeny trees. Describe evolutionary history of species.



3.2 Graph Traversal

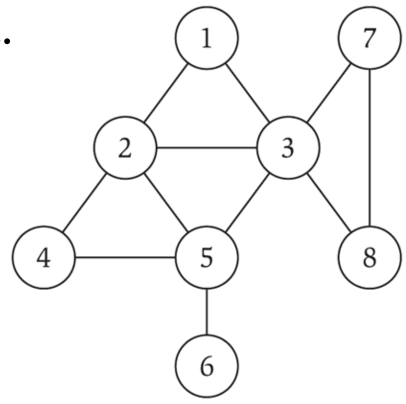
Connectivity

s-t connectivity problem. Given two nodes s and t, is there a path between s and t?

s-t shortest path problem. Given two node s and t, what is the length (number of edges) of the shortest path between s and t?

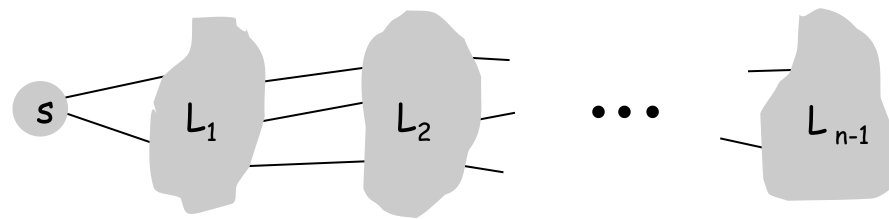
Applications.

- Facebook.
- Maze traversal.
- Erdos number.
- Fewest number of hops in a communication network.



Breadth First Search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



BFS algorithm.

- $L_0 = \{ s \}$.
- L_1 = all neighbours of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

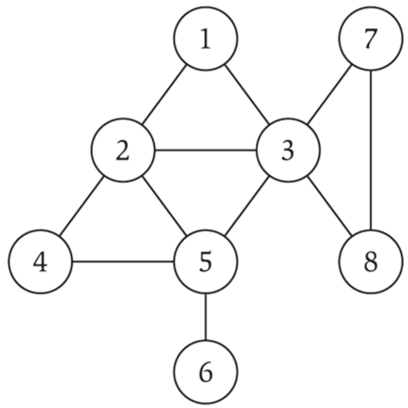
Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

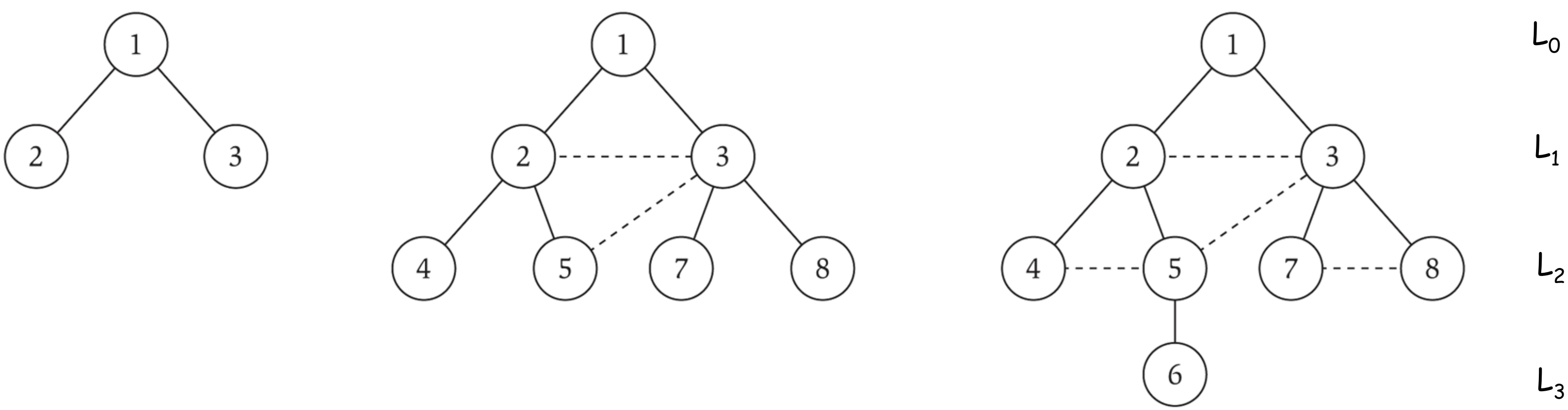
Breadth First Search

```
BFS(s):
  Set Discovered[s] = true and Discovered[v] = false for all other v
  Initialize L[0] to consist of the single element s
  Set the layer counter i = 0
  Set the current BFS tree T = Ø
  While L[i] is not empty
    Initialize an empty list L[i+1]
    For each node u ∈ L[i]
      Consider each edge (u, v) incident to u
      If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge (u, v) to the tree T
        Add v to the list L[i+1]
      Endif
    Endfor
    Increment the layer counter i by one
  Endwhile
```

Breadth First Search

Property. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G. Then the level of x and y differ by at most 1.





(a) (b) (c)

Breadth First Search: Analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency list representation.

Pf.

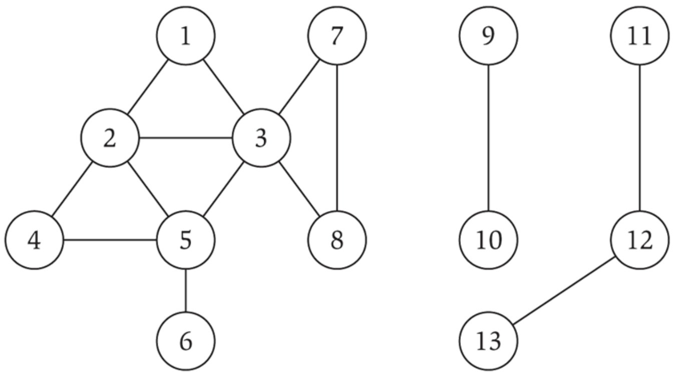
- Easy to prove $O(n^2)$ running time:
  - at most n lists L[i]
  - each node occurs on at most one list
  - when we consider node u, there are $\leq n$ incident edges (u, v), and we spend $O(1)$ processing each edge

- Actually runs in $O(m + n)$ time:
  - when we consider node u, there are deg(u) incident edges (u, v)
  - total time processing edges is $\Sigma_{u \in V} \deg(u) = 2m$ ▪

each edge (u, v) is counted exactly twice in sum: once in deg(u) and once in deg(v)

Connected Component

Connected component. Find all nodes reachable from s.



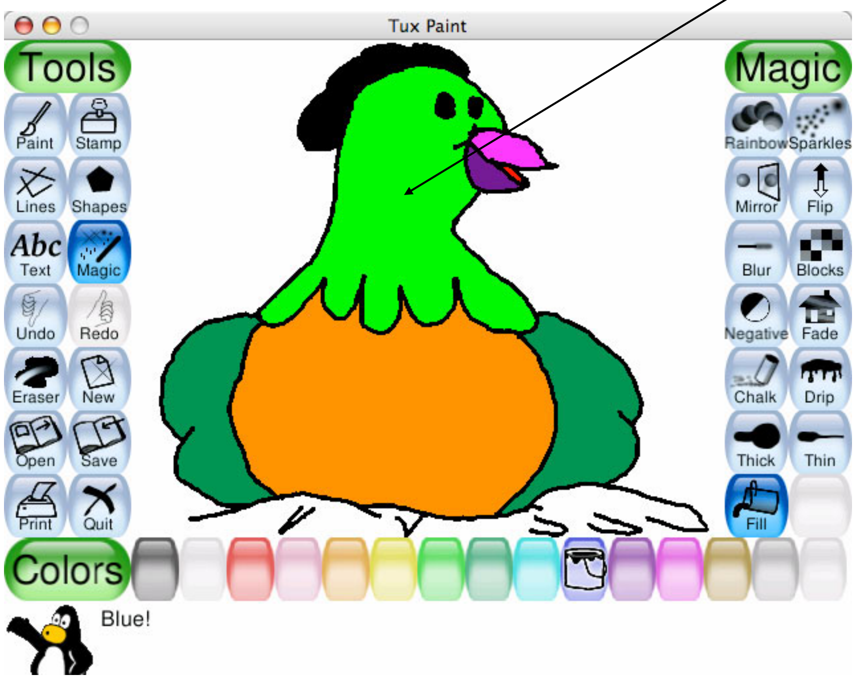
Connected component containing node 1 = { 1, 2, 3, 4, 5, 6, 7, 8 }.

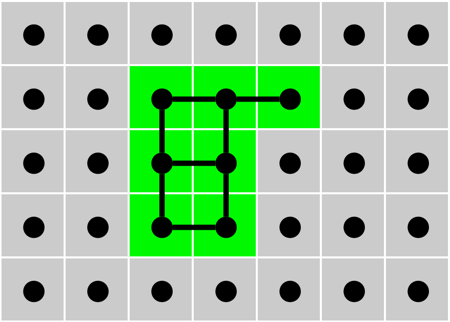
Flood Fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighbouring lime pixels to blue.

- Node: pixel.
- Edge: two neighbouring lime pixels.
- Blob: connected component of lime pixels.

recolour lime green blob to blue





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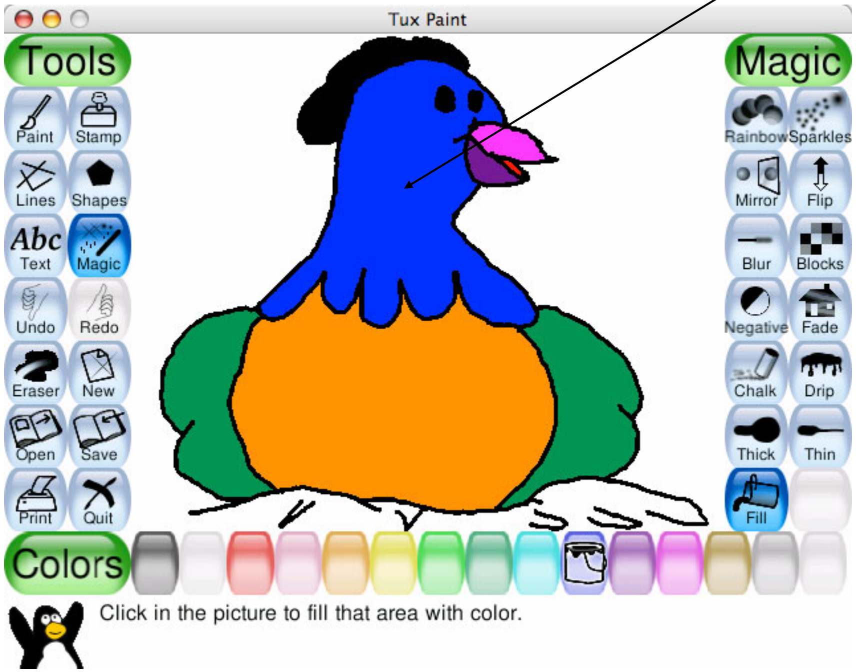
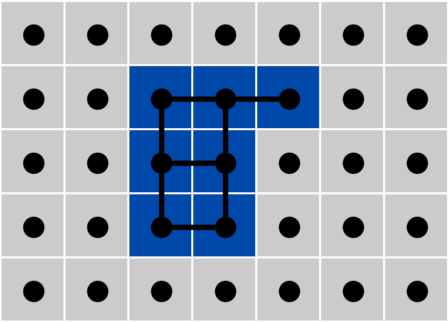
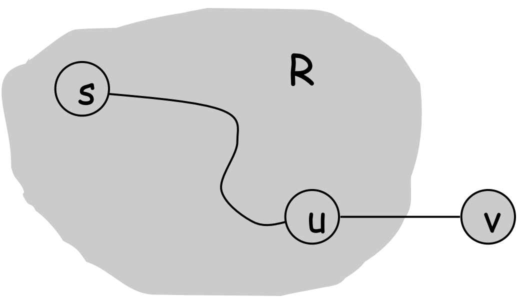




Connected Component

Connected component. Find all nodes reachable from s.

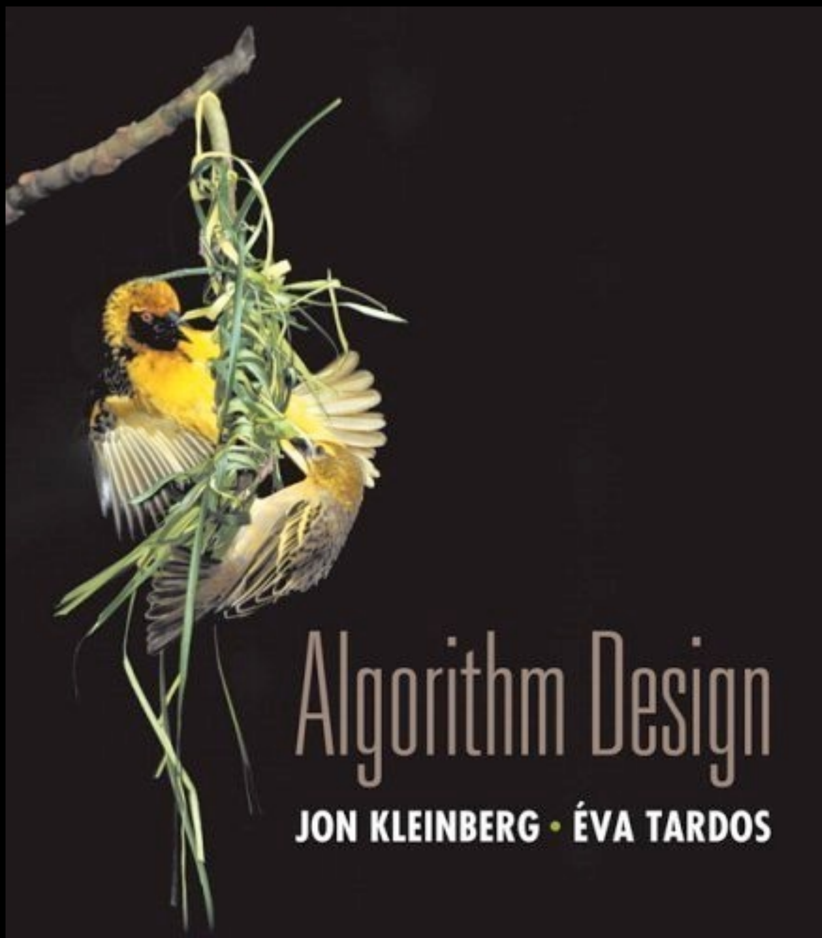
R will consist of nodes to which s has a path
Initially $R = \{s\}$
While there is an edge (u, v) where $u \in R$ and $v \notin R$
  Add v to R
Endwhile



it's safe to add v

Theorem. Upon termination, R is the connected component containing s.

- BFS = explore in order of distance from s.
- DFS = explore in a different way.



Chapter 3

Graphs

CLRS 12-13