Winter 2016
COMP-250: Introduction to Computer Science

Lecture 22, March 31, 2016

QuickSort

- Yet another sorting algorithm!
- Usually faster than other algorithms on average, although worst-case is $O(n^2)$
- Divide-and-conquer:
  - **Divide**: Choose an element of the array for *pivot*. Divide the elements into three groups: those smaller than the pivot, those equal, and those larger.
  - **Conquer**: Recursively sort each group.
  - **Combine**: Concatenate the three sorted groups.

QuickSort running time

- Worse case:
  - Already sorted array (either increasing or decreasing)
  - $T(n) = T(n-1) + c\, n + d$
  - $T(n)$ is $O(n^2)$
- Average case: If the array is in random order, the pivot splits the array in roughly equal parts, so the average running time is $O(n \log n)$
- Advantage over mergeSort:
  - constant hidden in $O(n \log n)$ are smaller for quickSort. Thus it is faster by a constant factor
  - QuickSort is easy to do "in-place"

In-place algorithms

- An algorithm is *in-place* if it uses only a *constant* amount of memory in addition of that used to store the input
- Importance of in-place sorting algorithms:
  - If the data set to sort barely fits into memory, we don't want an algorithm that uses twice that amount to sort the numbers
- SelectionSort and InsertionSort are in-place: all we are doing is moving elements around the array
- MergeSort is not in-place, because of the merge procedure, which requires a temporary array
- QuickSort can easily be made in-place...

Partition

Algorithm partition(A, start, stop)

Input: An array A, indices start and stop.

Output: Returns an index j and rearranges the elements of A
 such that for all i<j, A[i] ≤ A[j] and
 for all k>**j**, A[k] ≥ A[j].

pivot ← A[stop]

left ← start

right ← stop - 1

while left ≤ right **do**

  while left ≤ right **and** A[left] ≤ pivot) **do** left ← left + 1

  while (left ≤ right **and** A[right] ≥ pivot) **do** right ← right -1

  if (left < right **) then** exchange A[left] ↔ A[right]

exchange A[stop] ↔ A[left]

return left

Example of execution of partition

A = [6 3 7 3 2 5 7 5] pivot = 5

Example of execution of partition

A = [6 3 7 3 2 5 7 5] pivot = 5

A = [6 3 7 3 2 5 7 5] swap 6, 2

Example of execution of partition

A = [6 3 7 3 2 5 7 5] pivot = 5

A = [6 3 7 3 2 5 7 5] swap 6, 2

A = [2 3 7 3 6 5 7 5]

Example of execution of partition

A = [6 3 7 3 2 5 7 5]   pivot = 5

A = [6 3 7 3 2 5 7 5]   swap 6, 2

A = [2 3 7 3 6 5 7 5]

A = [2 3 7 3 6 5 7 5]   swap 7,3

Example of execution of partition

A = [6 3 7 3 2 5 7 5] pivot = 5

A = [6 3 7 3 2 5 7 5] swap 6, 2

A = [2 3 7 3 6 5 7 5]

A = [2 3 7 3 6 5 7 5] swap 7,3

A = [2 3 3 7 6 5 7 5]

Example of execution of partition

A = [6 3 7 3 2 5 7 5]   pivot = 5

A = [6 3 7 3 2 5 7 5]   swap 6, 2

A = [2 3 7 3 6 5 7 5]

A = [2 3 7 3 6 5 7 5]   swap 7,3

A = [2 3 3 7 6 5 7 5]

A = [2 3 3 7 6 5 7 5]   swap 7,pivot

A = [2 3 3 5 6 5 7 7]

≤5   ≥5

In-place quickSort

Algorithm quickSort(A, start, stop)

Input: An array A to sort, indices start and stop

Output: A[start...stop] is sorted

if (start < stop) **then**

  pivot ← partition(A, start, stop)

  quickSort(A, start, pivot-1)

  quickSort(A, pivot+1, stop)

Randomized Quicksort

```
RandomizedQuicksort(A,start,stop) {
   if |A| = 0 return

   choose a pivot A[i] uniformly at random (start ≤ i ≤ stop)
   exchange A[i] ↔ A[stop]

  pivot ← partition(A,start,stop)

   RandomizedQuicksort(A, start, pivot-1)
   RandomizedQuicksort(A, pivot+1, stop)
}
```

Quicksort

Running time.

- [Best case.] Select the median element as the pivot: quicksort makes $\Theta(n \log n)$ comparisons.
- [Worst case.] Select the smallest (or largest) element as the pivot: quicksort makes $\Theta(n^2)$ comparisons.

Randomize. Protect against worst case by choosing pivot at random.

Intuition. If we always select a pivot that is bigger than 25% of the elements and smaller than 25% of the elements, then quicksort makes $\Theta(n \log n)$ comparisons.

Notation. Label elements so that $x_1 < x_2 < \ldots < x_n$.

Randomized Quicksort: Expected Number of Comparisons

Theorem. Expected # of comparisons is $O(n \log n)$.

Theorem. [Knuth 1973] Stddev of number of comparisons is $\sim 0.65n$.

Ex. If $n = 1$ million, the probability that randomized quicksort takes less than $4n \ln n$ comparisons is at least 99.94%.

Chebyshev's inequality. $\Pr[|X - \mu| \geq k\delta] < 1 / k^2$.

Mean (μ) Stddev (δ)

Strings and Pattern Matching

- Brute Force,Rabin-Karp, Knuth-Morris-Pratt
- Regular Expressions

String Searching

- The object of string searching is to find the location of a specific text pattern within a larger body of text (e.g., a sentence, a paragraph, a book, etc.).
- As with most algorithms, the main considerations for string searching are speed and efficiency.
- There are a number of string searching algorithms in existence today, but the three we shall review are Brute Force,Rabin-Karp, and Knuth-Morris-Pratt.

Brute Force

- The Brute Force algorithm compares the pattern to the text, one character at a time, until unmatching characters are found:

```
TWO ROADS DIVERGED IN A YELLOW WOOD
ROADS
---------------------------------------
TWO ROADS DIVERGED IN A YELLOW WOOD
 ROADS
---------------------------------------
TWO ROADS DIVERGED IN A YELLOW WOOD
  ROADS
---------------------------------------
TWO ROADS DIVERGED IN A YELLOW WOOD
   ROADS
---------------------------------------
TWO ROADS DIVERGED IN A YELLOW WOOD
    ROADS
```

  - Compared characters are italicized.
  - Correct matches are in boldface type.

- The algorithm can be designed to stop on either the first occurrence of the pattern, or upon reaching the end of the text.

Brute Force

- The Brute Force algorithm compares the pattern to the text, one character at a time, until unmatching characters are found:

*T*WO ROADS DIVERGED IN A YELLOW WOOD
*R*OADS

T*W*O ROADS DIVERGED IN A YELLOW WOOD
 *R*OADS

This blank space is italicized and red

S DIVERGED IN A YELLOW WOOD

TWO ROADS DIVERGED IN A YELLOW WOOD
    *R*OADS

TWO ***ROADS*** DIVERGED IN A YELLOW WOOD
    ROADS

  - Compared characters are italicized.
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Brute Force

- The Brute Force algorithm compares the pattern to the text, one character at a time, until unmatching characters are found:

```
TWO ROADS DIVERGED IN A YELLOW WOOD
ROADS
```
```
TWO ROADS DIVERGED IN A YELLOW WOOD
 ROADS
```
```
TWO ROADS DIVERGED IN A YELLOW WOOD
  ROADS
```
```
TWO ROADS DIVERGED IN A YELLOW WOOD
   ROADS
```
```
TWO ROADS DIVERGED IN A YELLOW WOOD
    ROADS
```

  - Compared characters are italicized.
  - Correct matches are in boldface type.

- The algorithm can be designed to stop on either the first occurrence of the pattern, or upon reaching the end of the text.

Brute Force Pseudo-Code

- Here's the pseudo-code

```
do
   if (text letter == pattern letter)
      compare next letter of pattern to next
      letter of text
   else
      move pattern down text by one letter
while (entire pattern found or end of text)
```

```
cool cat Rolo went over the fence
cat
```

```
cool cat Rolo went over the fence
 cat
```

```
cool cat Rolo went over the fence
  cat
```

```
cool cat Rolo went over the fence
   cat
```

```
cool_cat Rolo went over the fence
    cat
```

```
cool cat Rolo went over the fence
     cat
```

Brute Force-Complexity

- Given a pattern M characters in length, and a text N characters in length...
- **Worst case**: compares pattern to each substring of text of length M. For example, M=5.
- This kind of case can occur for image data.

1) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
2) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
3) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
4) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAAH
AAAAH **5 comparisons made**
....
N) AAAAAAAAAAAAAAAAAAAAAAAAAA*AAAAH*
5 comparisons made *AAAAH*

- Total number of comparisons: M (N-M+1)
- Worst case time complexity: O(MN)

Brute Force-Complexity(cont.)

- Given a pattern M characters in length, and a text N characters in length...
- **Best case if pattern found**: Finds pattern in first M positions of text. For example, M=5.

1) *AAAAA*AAAAAAAAAAAAAAAAAAAAAAH
AAAAA **5 comparisons made**

- Total number of comparisons: M
- Best case time complexity: O(M)

Brute Force-Complexity(cont.)

- Given a pattern M characters in length, and a text N characters in length...
- **Best case if pattern not found**: Always mismatch on first character. For example, M=5.

1) *A*AAAAAAAAAAAAAAAAAAAAAAAAAH
 *O*OOOH **1 comparison made**

2) A*A*AAAAAAAAAAAAAAAAAAAAAAAAH
 *O*OOOH **1 comparison made**

3) AA*A*AAAAAAAAAAAAAAAAAAAAAAAH
 *O*OOOH **1 comparison made**

4) AAA*A*AAAAAAAAAAAAAAAAAAAAAAH
 *O*OOOH **1 comparison made**

...

N) AAAAAAAAAAAAAAAAAAAAAA*A*AAAH
 1 comparison made *O*OOOH

- Total number of comparisons: N
- Best case time complexity: O(N)

Rabin-Karp

- The Rabin-Karp string searching algorithm calculates a **hash value** for the pattern, and for each M-character subsequence of text to be compared.
- If the hash values are unequal, the algorithm will calculate the hash value for next M-character sequence.
- If the hash values are equal, the algorithm will do a Brute Force comparison between the pattern and the M-character sequence.
- In this way, there is only one comparison per text subsequence, and Brute Force is only needed when hash values match.
- Perhaps an example will clarify some things...

Rabin-Karp Example

Hash value of "AAAAA" is 37

Hash value of "AAAAH" is 100

1) AAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
37≠100 **1 comparison made**

2) AAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
37≠100 **1 comparison made**

3) AAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
37≠100 **1 comparison made**

4) AAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
37≠100 **1 comparison made**

...

N) AAAAAAAAAAAAAAAAAAAAAAAAAAAH
AAAAH
5 comparisons made 100=100

Rabin-Karp Algorithm

pattern is M characters long

hash_p=hash value of pattern
hash_t=hash value of first M letters in
body of text

```
do
  if (hash_p == hash_t)
     brute force comparison of pattern
     and selected section of text
  hash_t = hash value of next section of
           text, one character over
until (end of text or
       brute force comparison == true)
```

Rabin-Karp

- Common Rabin-Karp questions:

  "What is the hash function used to calculate values for character sequences?"

  "Isn't it time consuming to hash every one of the M-character sequences in the text body?"

  "Is this going to be on the final?"

- To answer some of these questions, we'll have to get mathematical.

Rabin-Karp Math

- Consider an M-character sequence as an M-digit number in base b, where b is the number of letters in the alphabet. The text subsequence t[i .. i+M-1] is mapped to the number

$$x(i) = t[i] \cdot b^{M-1} + t[i+1] \cdot b^{M-2} + \ldots + t[i+M-1]$$

- Furthermore, given x(i) we can compute x(i+1) for the next subsequence t[i+1 .. i+M] in constant time, as follows:

$$x(i+1) = t[i+1] \cdot b^{M-1} + t[i+2] \cdot b^{M-2} + \ldots + t[i+M]$$

$x(i+1) = x(i) \cdot b$ Shift left one digit

$- t[i] \cdot b^{M}$ Subtract leftmost digit

$+ t[i+M]$ Add new rightmost digit

- In this way, we never explicitly compute a new value. We simply adjust the existing value as we move over one character.

Rabin-Karp Math Example

- Let’s say that our alphabet consists of 10 letters.
- our alphabet = a, b, c, d, e, f, g, h, i, j
- Let’s say that “a” corresponds to 1, “b” corresponds to 2 and so on.

The hash value for string “cah” would be ...

3*100 + 1*10 + 8*1 = 318

Rabin-Karp Mods

- If M is large, then the resulting value (~bM) will be enormous. For this reason, we hash the value by taking it **mod** a prime number ***q***.
- The **mod** function (% in Java) is particularly useful in this case due to several of its inherent properties:
  - [(x mod q) + (y mod q)] mod q = (x+y) mod q
  - (x mod q) mod q = x mod q

Rabin-Karp Mods

- For these reasons:

$$\boldsymbol{h}(i) = ((\boldsymbol{t}[i] \cdot \boldsymbol{b}^{M-1} \bmod \boldsymbol{q}) + (\boldsymbol{t}[i+1] \cdot \boldsymbol{b}^{M-2} \bmod \boldsymbol{q}) + \ldots + (\boldsymbol{t}[i+M-1] \bmod \boldsymbol{q})) \bmod \boldsymbol{q}$$

$\boldsymbol{h}(i+1) = (\boldsymbol{h}(i) \cdot \boldsymbol{b} \bmod \boldsymbol{q}$

Shift left one digit

$-\boldsymbol{t}[i] \cdot \boldsymbol{b}^{M} \bmod \boldsymbol{q}$

Subtract leftmost digit

$+\boldsymbol{t}[i+M] \bmod \boldsymbol{q}$)

Add new rightmost digit

$\bmod \boldsymbol{q}$

Rabin-Karp Complexity

- If a sufficiently large prime number is used for the *hash function*, the hashed values of two different patterns will usually be distinct.
- If this is the case, searching takes O(N) time, where N is the number of characters in the larger body of text.
- It is always possible to construct a scenario with a worst case complexity of O(MN). This, however, is likely to happen only if the prime number used for hashing is small.

The Knuth-Morris-Pratt Algorithm

- The Knuth-Morris-Pratt (KMP) string searching algorithm differs from the brute-force algorithm by keeping track of information gained from previous comparisons.

- A failure function (*f*) is computed that indicates how much of the last comparison can be reused if it fails.

- Specifically, *f* is defined to be the longest prefix of the pattern P[0,..,j] that is also a suffix of P[1,..,j]
  - Note: **not** a suffix of P[0,..,j]

The Knuth-Morris-Pratt Algorithm

- Specifically, *f* is defined to be the longest prefix of the pattern P[0,..,j] that is also a suffix of P[1,..,j]
  - Note: **not** a suffix of P[0,..,j]
- Example:
  - value of the KMP failure function:

j	0	1	2	3	4	5
$P[j]$	a	b	a	b	a	c
$f(j)$	0	0	1	2	3	0

- This shows how much of the beginning of the string matches up to the portion immediately preceding a failed comparison.
  - if the comparison fails at (4), we know the a,b in positions 2,3 is identical to positions 0,1

The KMP Algorithm (contd.)

- the KMP string matching algorithm: Pseudo-Code

```
Algorithm KMPMatch(T,P)
  Input: Strings T (text) with n characters and P
     (pattern) with m characters.
  Output: Starting index of the first substring of T
     matching P, or an indication that P is not a
     substring of T.

  f ← KMPFailureFunction(P) {build failure function}
  i ← 0
  j ← 0
  while i < n do
     if P[j] = T[i] then
        if j = m - 1 then
           return i - m - 1 {a match}
        i ← i + 1
        j ← j + 1
     else if j > 0 then {no match, but we have advanced}
        j ← f(j-1) {j indexes just after matching prefix in P}
     else
        i ← i + 1
  return "There is no substring of T matching P"
```

The KMP Algorithm (contd.)

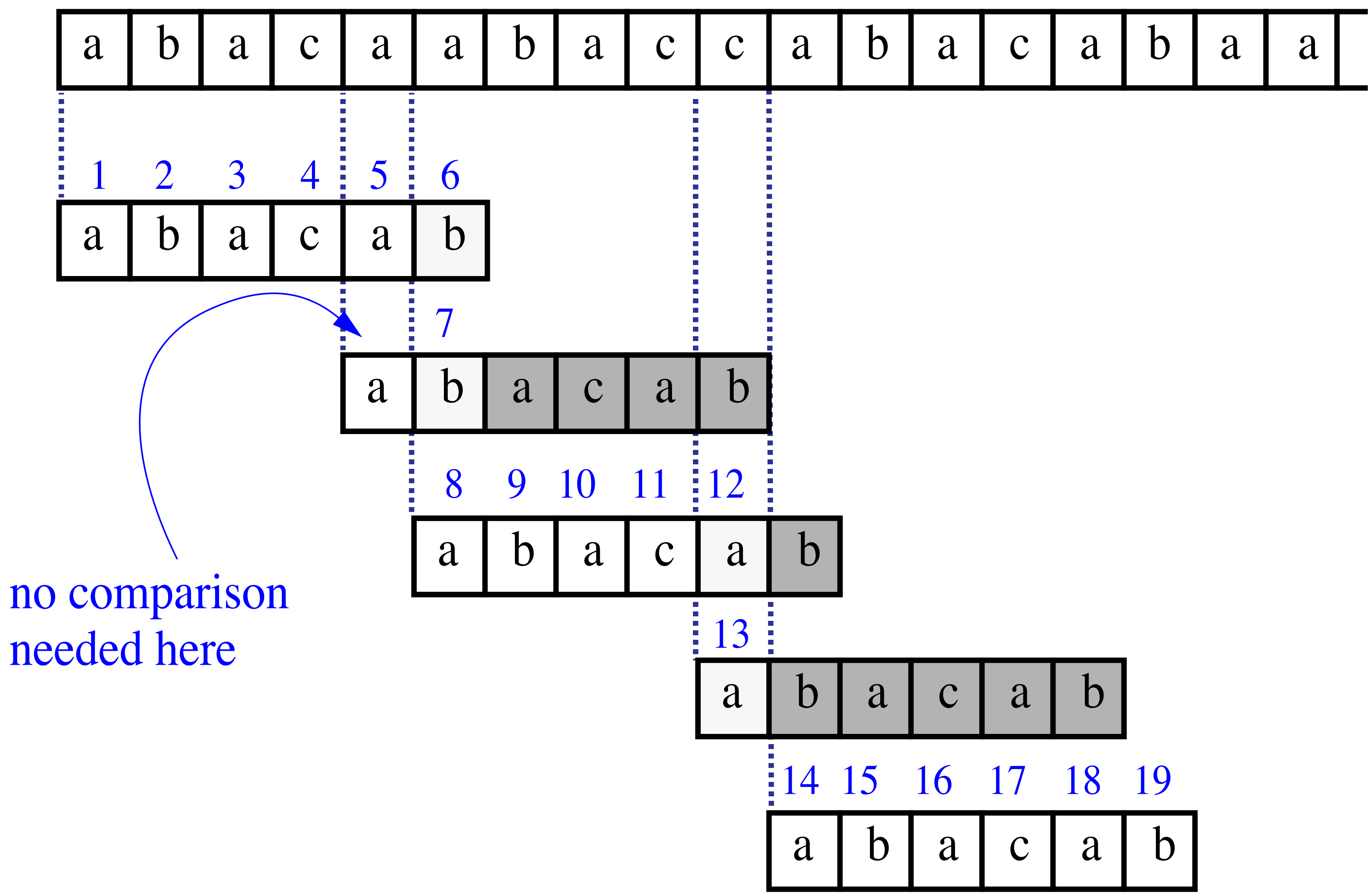
- The KMP failure function: Pseudo-Code

Algorithm KMPFailureFunction(P);
Input: String P (pattern) with m characters
Ouput: The faliure function f for P, which maps j to the length of the longest prefix of P that is a suffix of $P[1,..,j]$

```
i ← 1
j ← 0
while i ≤ m-1 do
   if P[j] = P[i] then
      {we have matched j + 1 characters}
      f(i) ← j + 1
      i ← i + 1
      j ← j + 1
   else if j > 0 then
      {j indexes just after a prefix of P that matches}
      j ← f(j-1)
   else
      {there is no match}
      f(i) ← 0
      i ← i + 1
```

The KMP Algorithm (contd.)

- A graphical representation of the KMP string searching algorithm



The KMP Algorithm (contd.)

- Time Complexity Analysis
- define $k = i - j$
- In every iteration through the while loop, one of three things happens.
  - 1) if $T[i] = P[j]$, then i increases by 1, as does j
    k remains the same.
  - 2) if $T[i] \mathrel{!}= P[j]$ and $j > 0$, then i does not change and k increases by at least 1, since k changes from $i - j$ to $i - f(j-1)$
  - 3) if $T[i] \mathrel{!}= P[j]$ and $j = 0$, then i increases by 1 and k increases by 1 since j remains the same.

The KMP Algorithm (contd.)

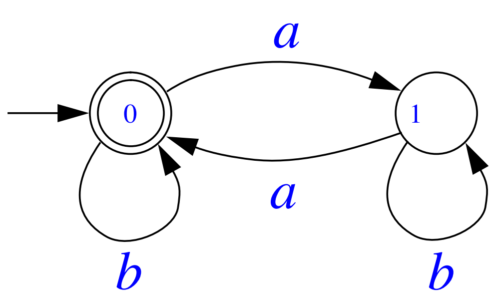
- Thus, each time through the loop, either i or k increases by at least 1, so the greatest possible number of loops is $2n$
- This of course assumes that f has already been computed.
- However, f is computed in much the same manner as KMPMatch so the time complexity argument is analogous. KMPFailureFunction is $\boldsymbol{O}(m)$
- Total Time Complexity: $\boldsymbol{O}(n + m)$

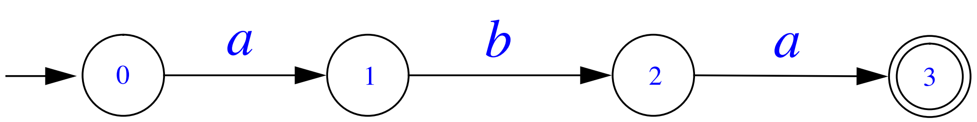
Regular Expressions

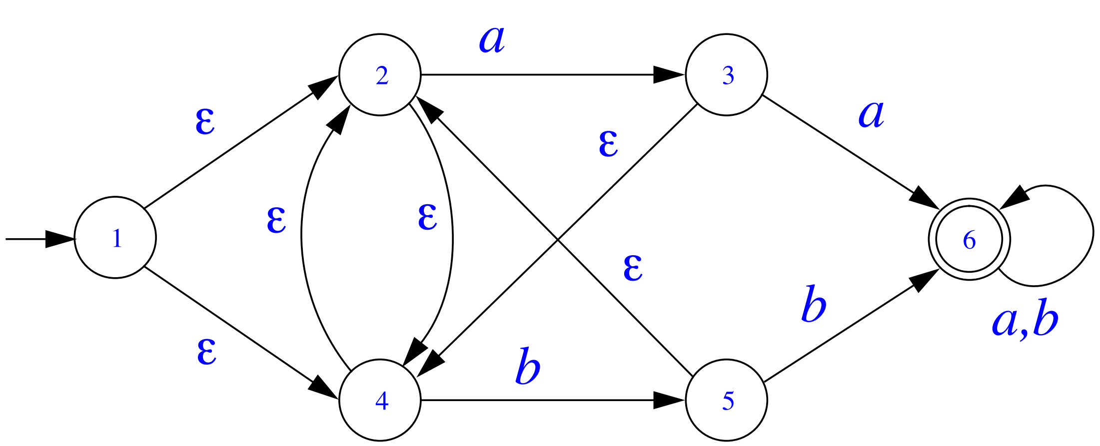
- notation for describing a set of strings, possibly of infinite size
- ε denotes the empty string
- **ab + c** denotes the set {ab, c}
- **a*** denotes the set {ε, a, aa, aaa, ...}
- Examples
  - **(a+b)*** all the strings from the alphabet {a,b}
  - **b*(ab*a)*b*** strings with an even number of a's
  - **(a+b)*sun(a+b)*** strings containing the pattern "sun"
  - **(a+b)(a+b)(a+b)a** 4-letter strings ending in a

Finite State Automaton

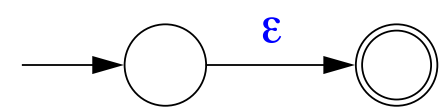
- “machine” for processing strings

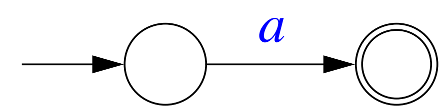


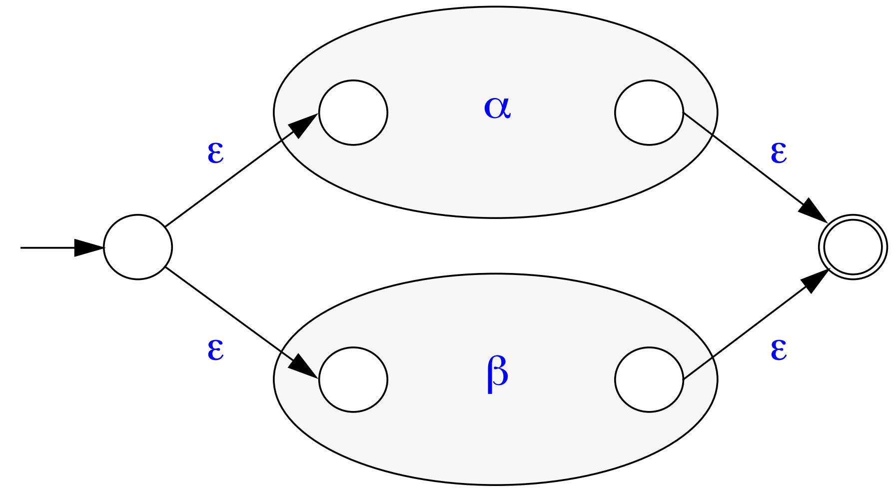


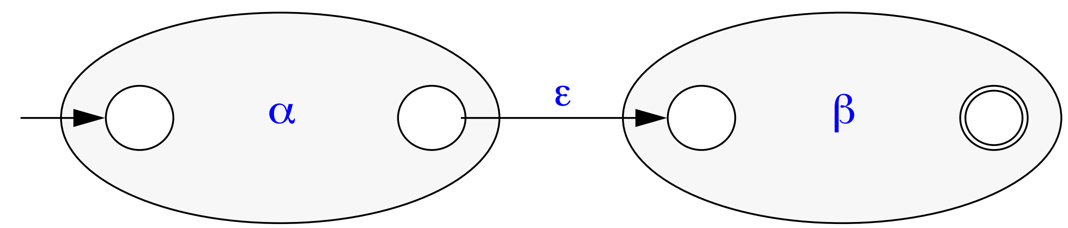


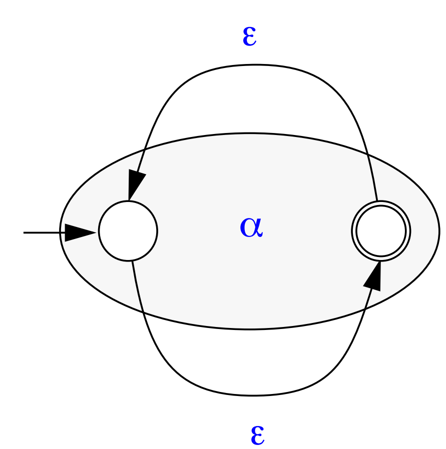
Composition of FSA’s











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