COMP 102A, Lecture 12

Computability and Complexity

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- Speaking on 8 August 1900, at the Paris 2nd International Congress of Mathematicians, at La Sorbonne. The full list was published later.
- The problems were all unsolved at the time, and several of them turned out to be very influential for 20th century mathematics.

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- Is this just because we are not smart enough to find a solution ?
- Or, is there something deeper going on ?

computer science version of these questions

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- If my boss / supervisor / teacher formulates a problem to be solved urgently, can I write a program to solve this problem in an efficient manner ???

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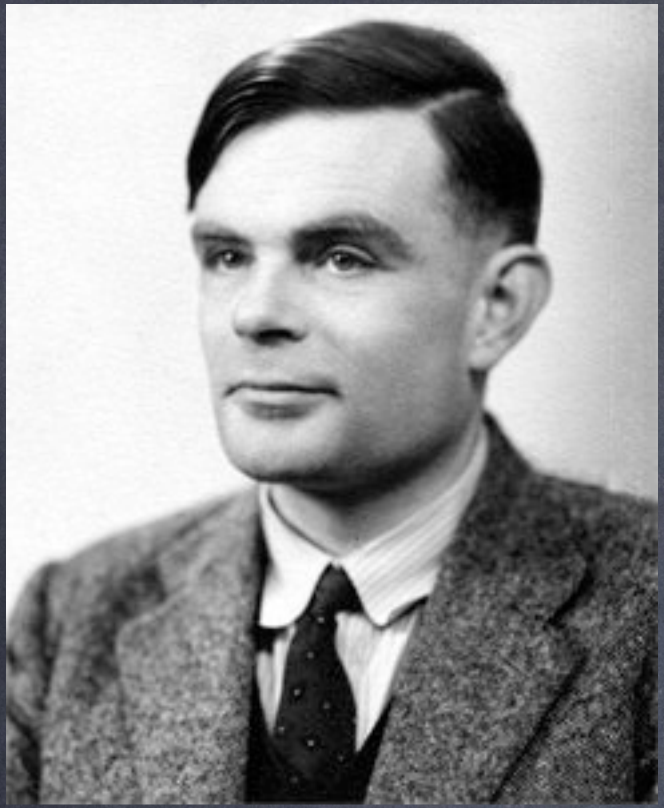
- If my boss / supervisor / teacher formulates a problem to be solved urgently, can I write a program to solve this problem in an efficient manner ???
- Are there some problems that cannot be solved at all ? and, are there problems that cannot be solved efficiently ??
(related to Hilbert's 10^{th} problem)

Kurt Gödel

Kurt Gödel

- In 1931, he proved that any formalization of mathematics contains some statements that cannot be proved or disproved.

Alan Turing



Alan Turing

- In 1934, he formalized the notion of **decidability** of a language by a computer.

A Language

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- An algorithm decides a language if it answers Yes when x is in L and No otherwise

Comparing Cardinalities

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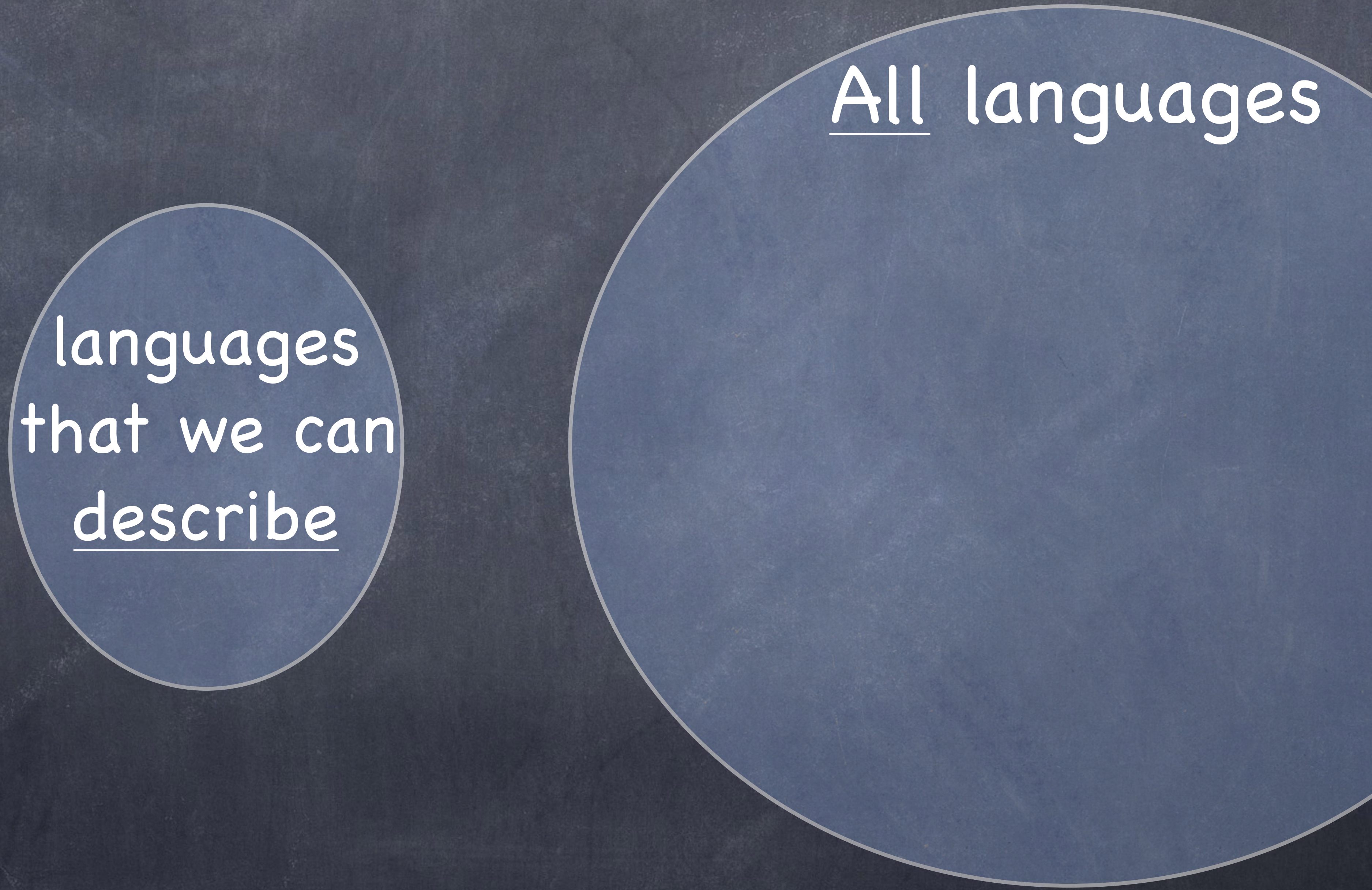

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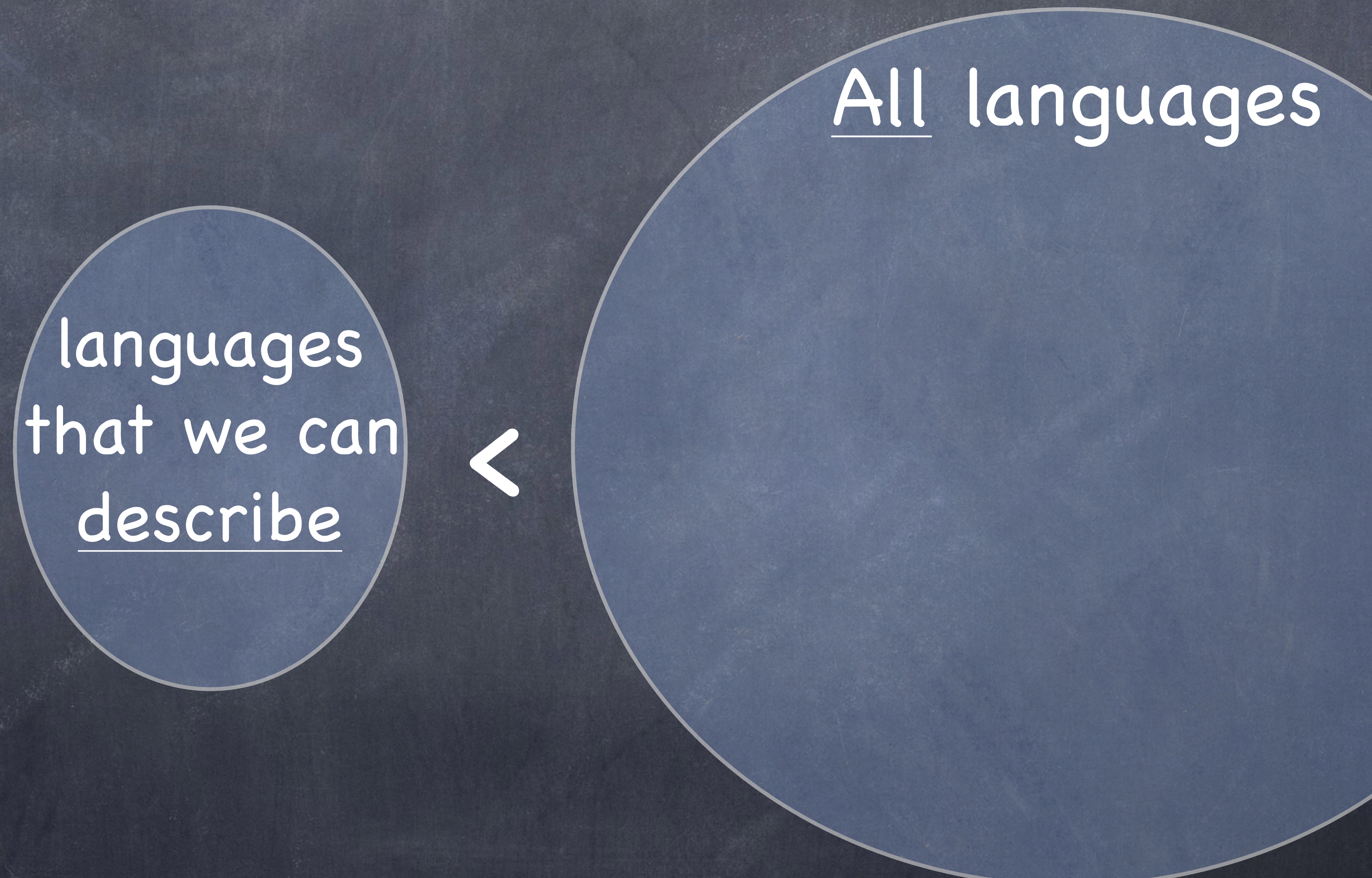

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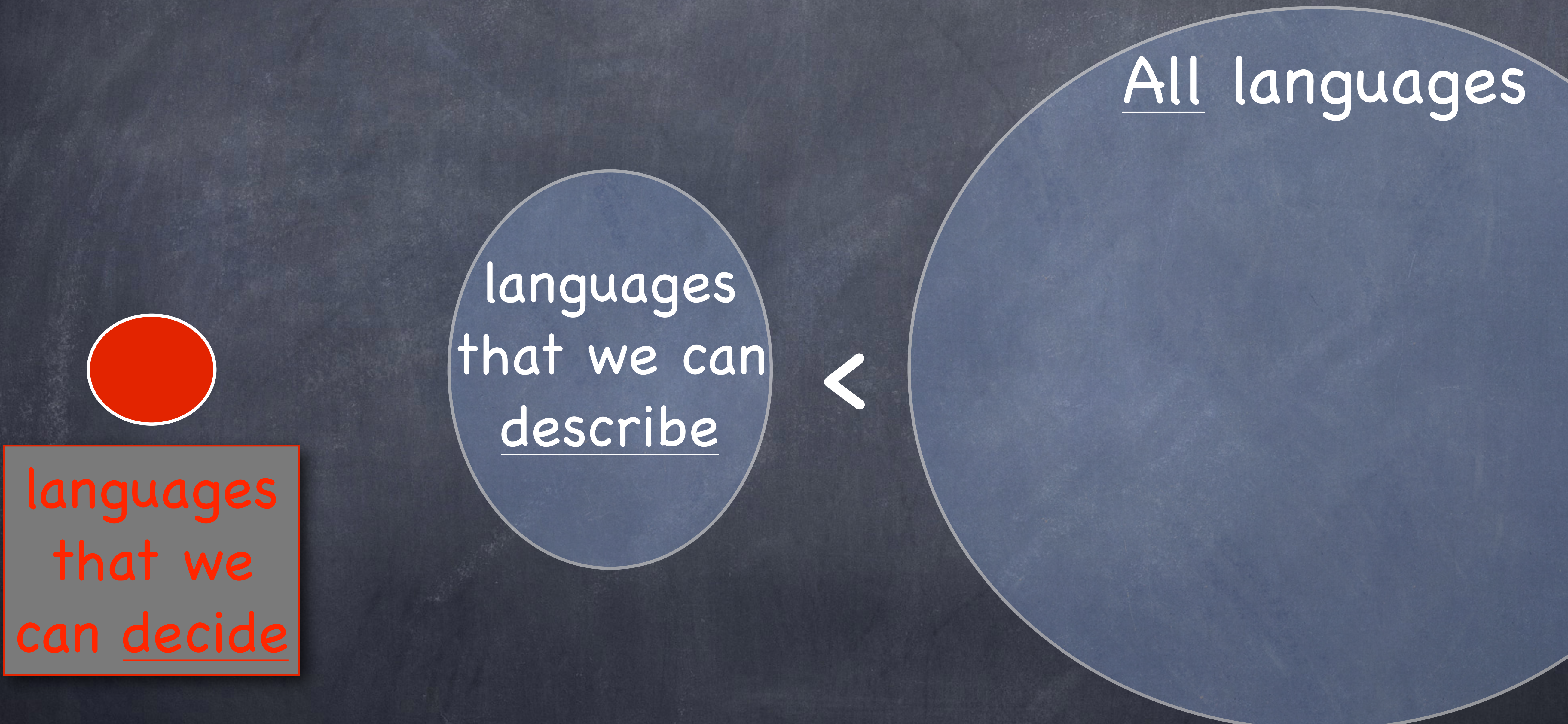

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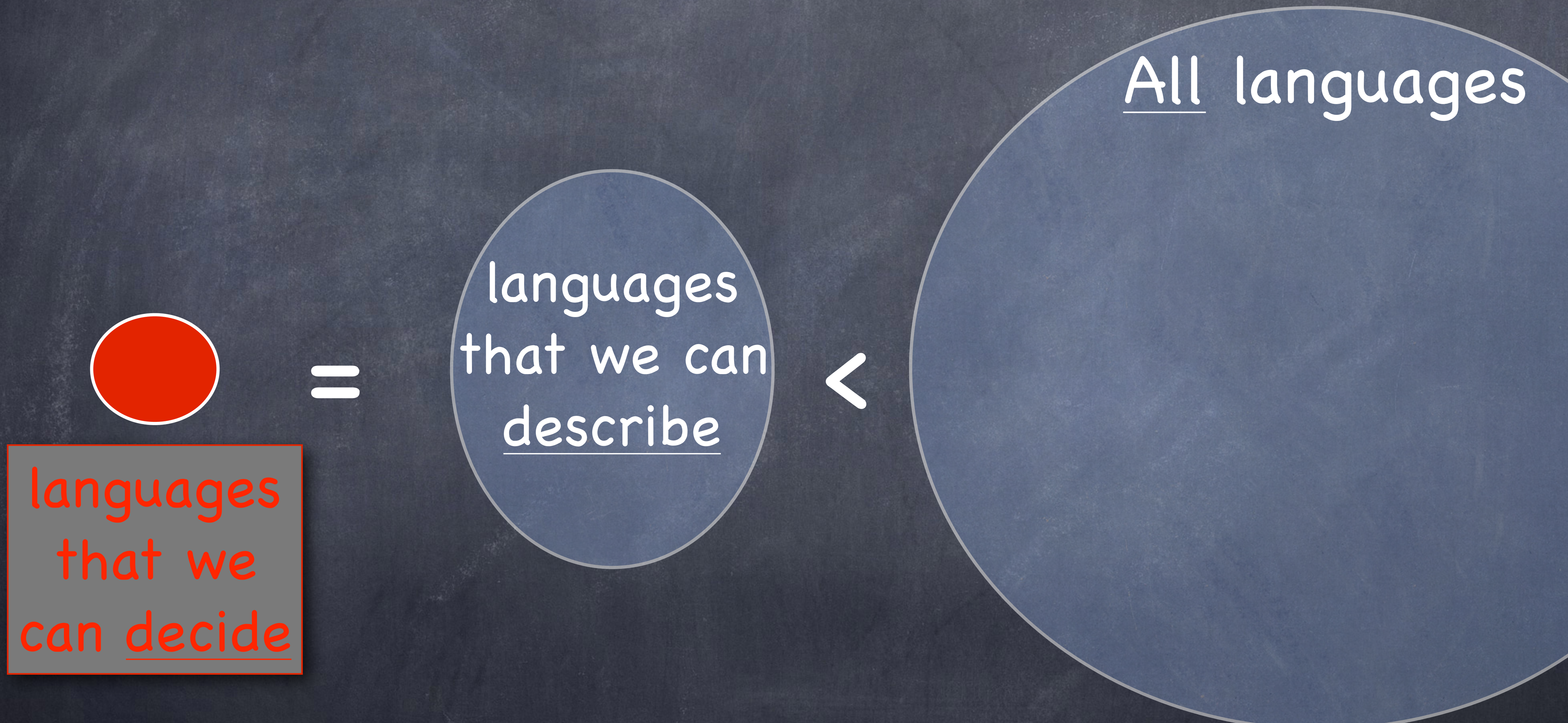

Alonzo Church

Alonzo Church

- In 1936, he proved that certain **languages** cannot be **decided** by any algorithm whatsoever...

Emil Post

Emil Post

- In 1946, he gave a very natural example of an **undecidable** language...

(**PCP**) Post Correspondence Problem

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aaa	a	bbb	aa		b
bb	bb	a	a	bb	

- An instance of **PCP** with 6 tiles.

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Post
Correspondence Problem

Post Correspondence Problem

u_1	u_2	u_3	. . .	u_n
v_1	v_2	v_3		v_n

- Given n tiles, u_1/v_1 … u_n/v_n
  where each u_i or v_i is a sequence of letters.

Post Correspondence Problem

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- Given n tiles, u_1/v_1 … u_n/v_n
  where each u_i or v_i is a sequence of letters.
- Is there a k and a sequence $\langle i_1, i_2, i_3, \ldots, i_k \rangle$
  (with each $1 \le i_j \le n$) such that

$$u_{i_1} \mid u_{i_2} \mid u_{i_3} \mid \ldots \mid u_{i_k} = v_{i_1} \mid v_{i_2} \mid v_{i_3} \mid \ldots \mid v_{i_k} ?$$

A Solution to Post Correspondence Problem

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- A solution is of this form:
  with the top and bottom strings identical.

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- A solution is of this form:
  with the top and bottom strings identical.

u_{i_1}	u_{i_2}	u_{i_3}	u_{i_4}	u_{i_5}	. . .	u_{i_k}
v_{i_1}	v_{i_2}	v_{i_3}	v_{i_4}	v_{i_5}		v_{i_k}

Post
Correspondence Problem

Post Correspondence Problem

- **Theorem:**

  The Post Correspondence Problem cannot be **decided** by any algorithm (or computer program). In particular, no algorithm can identify in a finite amount of time the instances that have a negative outcome. However, if a solution exists, we can find it.

Post
Correspondence Problem

Post Correspondence Problem

- **Proof**:

  Reduction technique - if **PCP** was **decidable** then another **undecidable** problem would be **decidable**.

The Halting Problem

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- The Halting Problem:
  Given two texts A,B, consider A as an algorithm and B as an input. Will algorithm A halt (as opposed to loop forever) on input B?

The Halting Problem

The Halting Problem

- **Theorem**: no algorithm can decide the Halting Problem.

The Halting Problem

- **Theorem**: no algorithm can decide the Halting Problem.
- **Proof**: Assume for a contradiction that an algorithm **Halt(A,B)** exists to decide the Halting Problem.

The Halting Problem

The Halting Problem

- Consider the Algorithm:

  Bug(A)

  if Halt(A,A) then While True do

  { when **Halt(A,A)** is true then **Bug(A)** loops }

  { when **Halt(A,A)** is false then **Bug(A)** halts }

The Halting Problem

- Consider the Algorithm:

  Bug(A)

  if Halt(A,A) then While True do

  { when **Halt(A,A)** is true then **Bug(A)** loops }

  { when **Halt(A,A)** is false then **Bug(A)** halts }

- Question: What is the outcome of **Bug(Bug)**?

The Halting Problem

The Halting Problem

- If **Bug(Bug)** does not loop forever it is because **Halt(Bug,Bug)=False** which means **Bug(Bug)** loops forever. (contradiction)

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- If **Bug(Bug)** does not loop forever it is because **Halt(Bug,Bug)=False** which means **Bug(Bug)** loops forever. (contradiction)
- If **Bug(Bug)** loops forever it is because **Halt(Bug,Bug)=True** which means **Bug(Bug)** does not loop forever. (contradiction)
- Conclusion: **Halt** cannot exist.

The Halting Problem and **PCP**

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- Any algorithm to decide **PCP** can be converted to an algorithm to decide the Halting Problem.

The Halting Problem and **PCP**

- Any algorithm to decide **PCP** can be converted to an algorithm to decide the Halting Problem.
- **Conclusion**: **PCP** cannot be decided either.

Computability Theory

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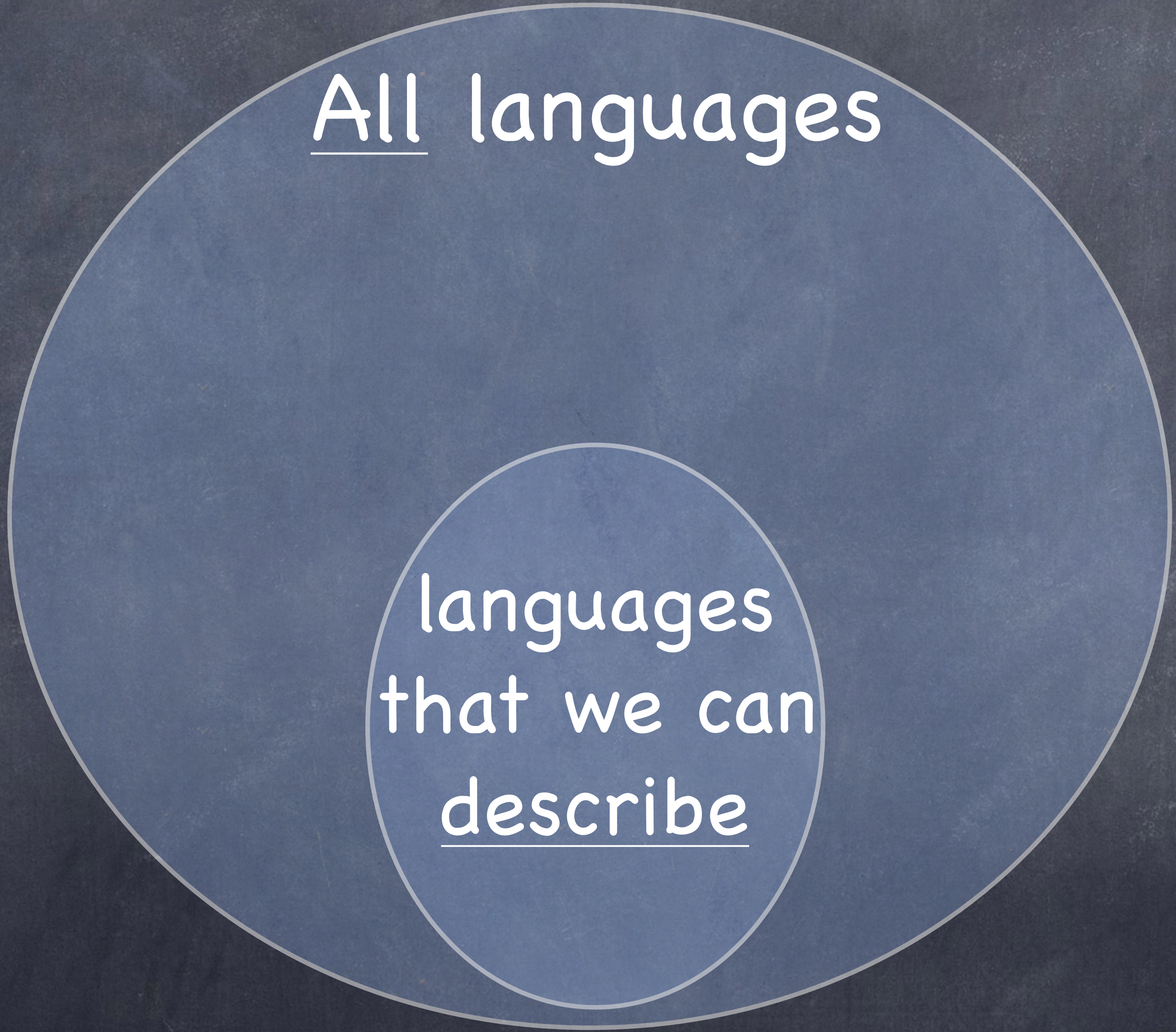
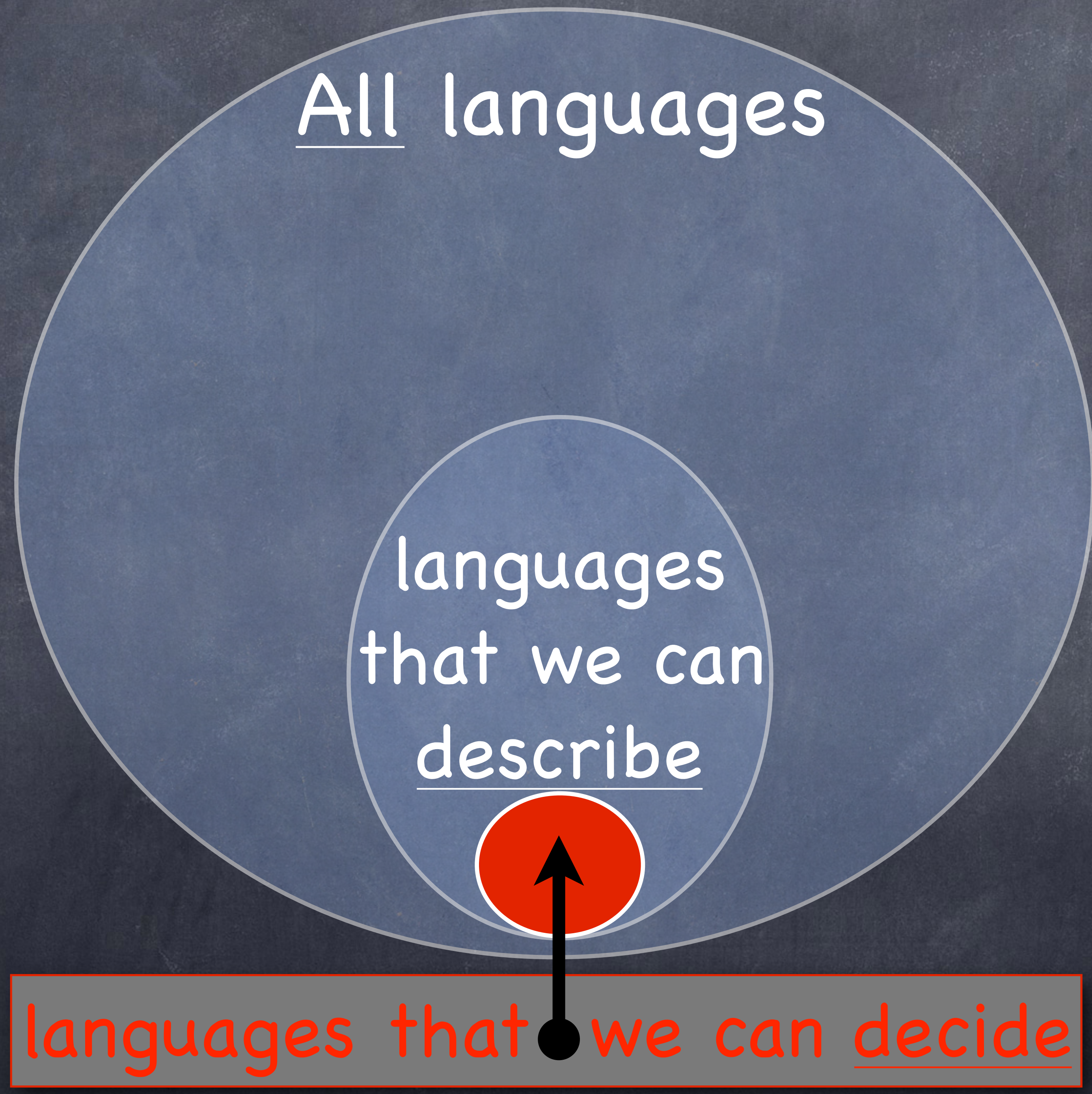


Computability Theory



COMP 102A 2013

Decidable ? Some times we just don’t know…

COMP 102A 2013

Syracuse Conjecture

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Syracuse Conjecture

- For any integer n>0 define the following sequence:

$$S_1=n,\ S_{i+1}=\begin{cases} S_i/2 & \text{if } S_i \text{ is even,} \\ 3S_i+1 & \text{if } S_i \text{ is odd.} \end{cases}$$

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Syracuse Conjecture

- Example: **Syracuse**(9) = 20

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- S_1=9, S_2=28, S_3=14, S_4=7, S_5=22, S_6=11, S_7=34, S_8=17, S_9=52, S_{10}=26, S_{11}=13, S_{12}=40, S_{13}=20, S_{14}=10, S_{15}=5, S_{16}=16, S_{17}=8, S_{18}=4, S_{19}=2, S_{20}=1

Syracuse Conjecture

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- For all n that we have computed so far, **Syracuse**(n) > 0.

Syracuse Conjecture

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for all n>0,   Syracuse(n)>0

Syracuse Conjecture

- For all n that we have computed so far, **Syracuse**
- **<u>Conjecture</u>**

  for all n>0, Syracuse(n)>0

- If there exists N such that **Syracuse**(N) = 0 we might not be able to prove it.

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- Even worse, it might be decidable but there might be no proof that it is !!!

Complexity and Tractability

COMP 102A, Lecture 13

Not all problems were born equal…

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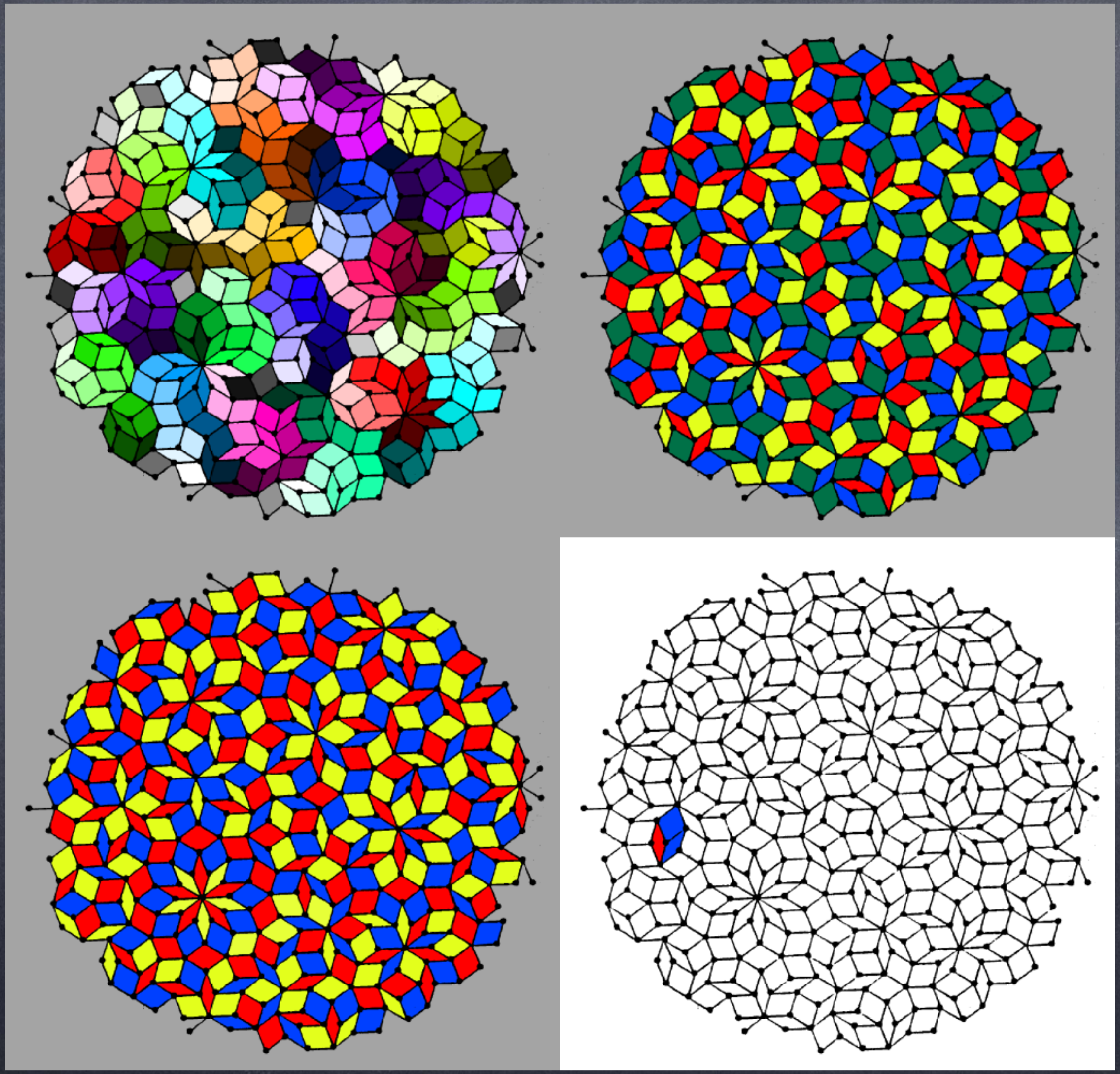
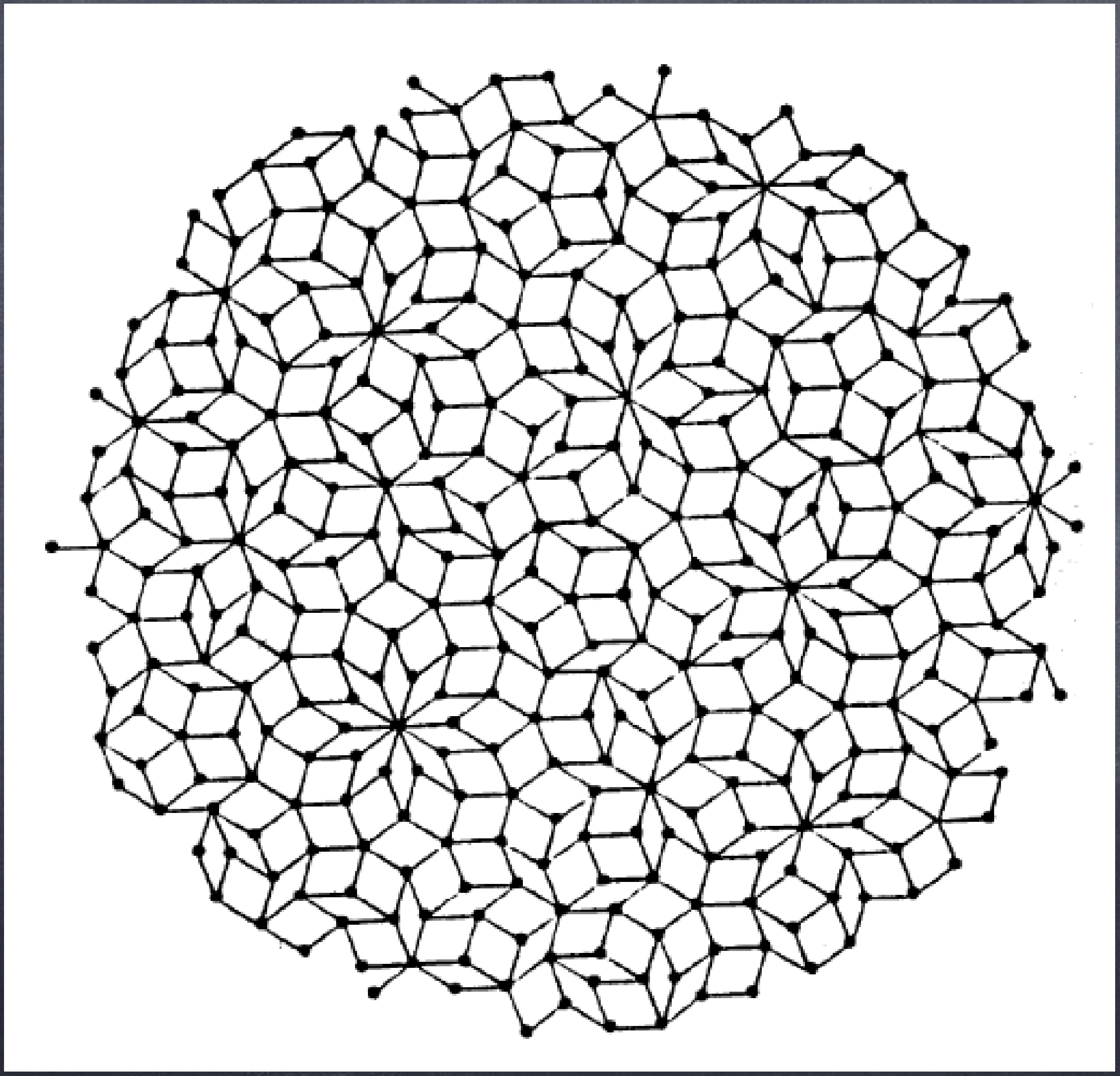
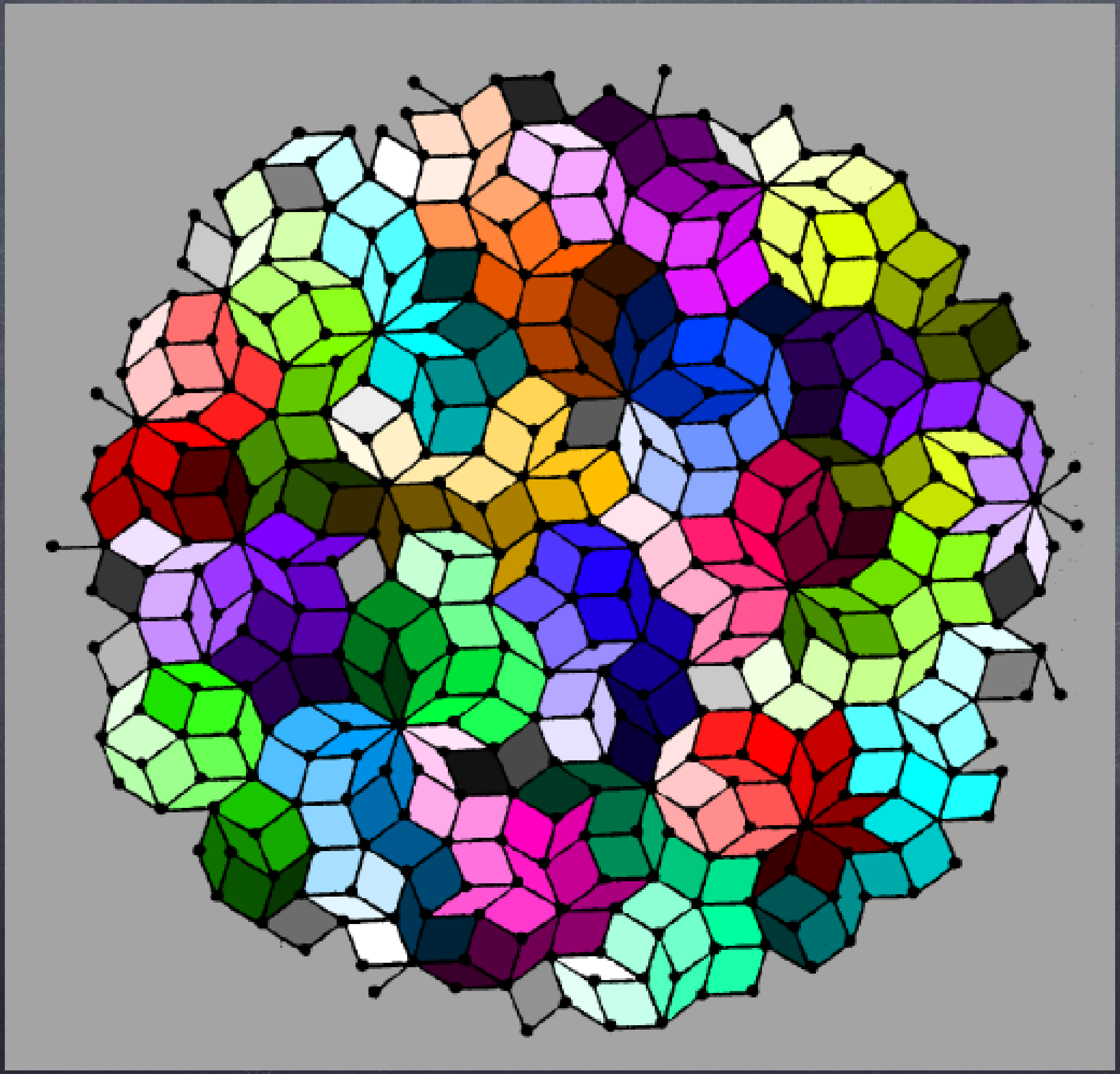


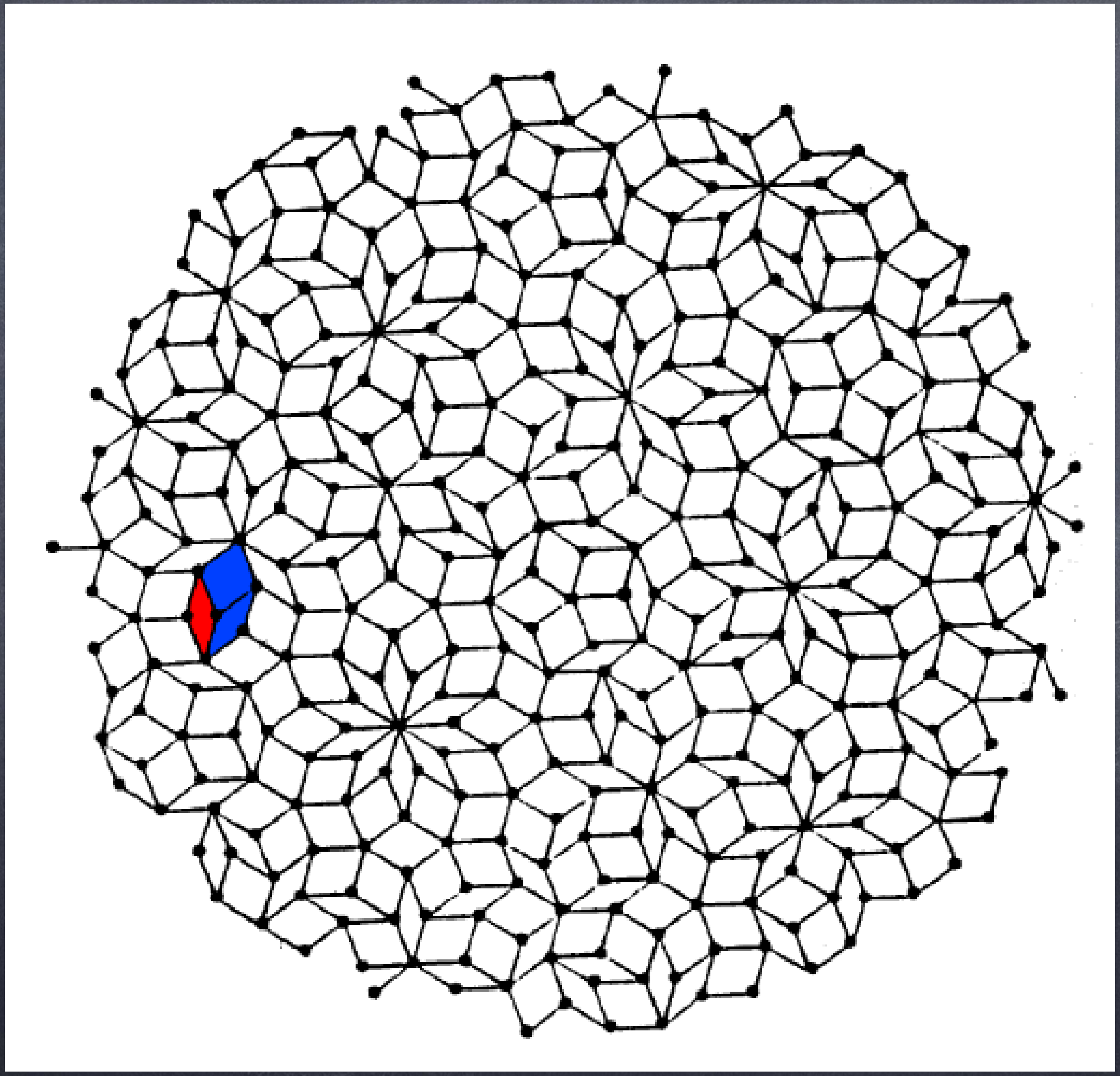
Is it possible to paint a colour on each region of a map so that no neighbours are of the same colour ?



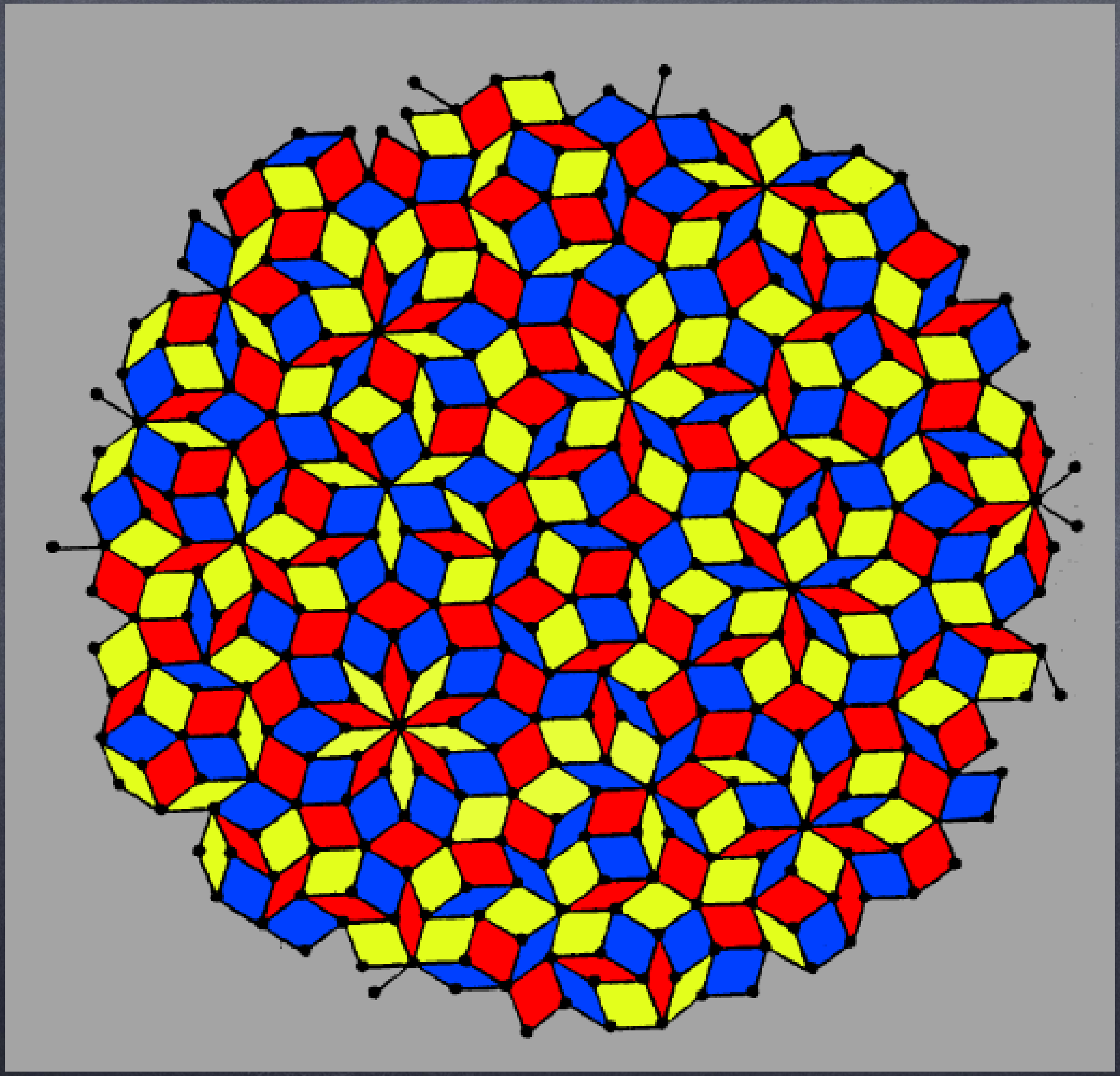
Obviously, yes, if you can use as many colours as you like...



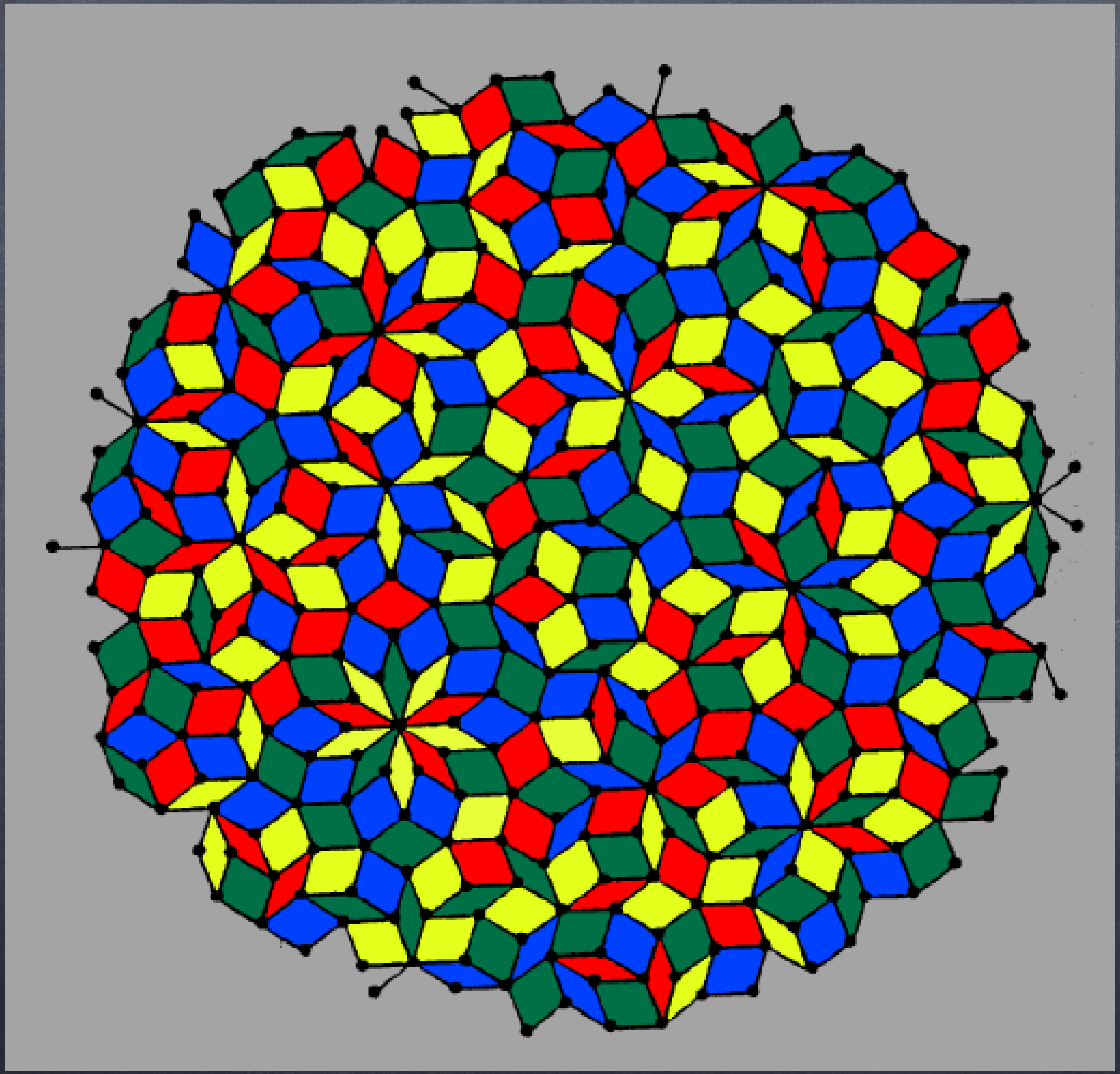
2 colouring problem



3 colouring problem



4 colouring problem



K-colouring of Maps (planar graphs)

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K-colouring of Maps (planar graphs)

- K=1, only the map with zero or one region are 1-colourable.
- K=2, easy to decide. Impossible as soon as 3 regions touch each other.
- K=3, No known efficient algorithm to decide. However it is easy to verify a solution.
- K≥4, all maps are K-colourable. (hard proof) Does not imply easy to find a K-colouring.

3-colouring of Maps

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- Seems hard to solve in general,

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- Is easy to verify when a solution is given,

3-colouring of Maps

- Seems hard to solve in general,
- Is easy to verify when a solution is given,
- Is a special type of problem (NP-complete) because an efficient solution to it would yield efficient solutions to MANY similar problems !

Examples of NP-Complete Problems

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- SAT: given a boolean formula, is there an assignment of the variables making the formula evaluate to true ?

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- SAT: given a boolean formula, is there an assignment of the variables making the formula evaluate to true ?
- Travelling Salesman: given a set of cities and distances between them, what is the shortest route to visit each city once.
- KnapSack: given items with various weights, is there of subset of them of total weight K.

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- In practice, some of them may be solved efficiently in some special cases.

Tractable Problems (P)

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- Computer Science studies mostly techniques to approach and find efficient solutions to tractable problems.
- Some problems may be efficiently solvable but we might not be able to **prove** that...

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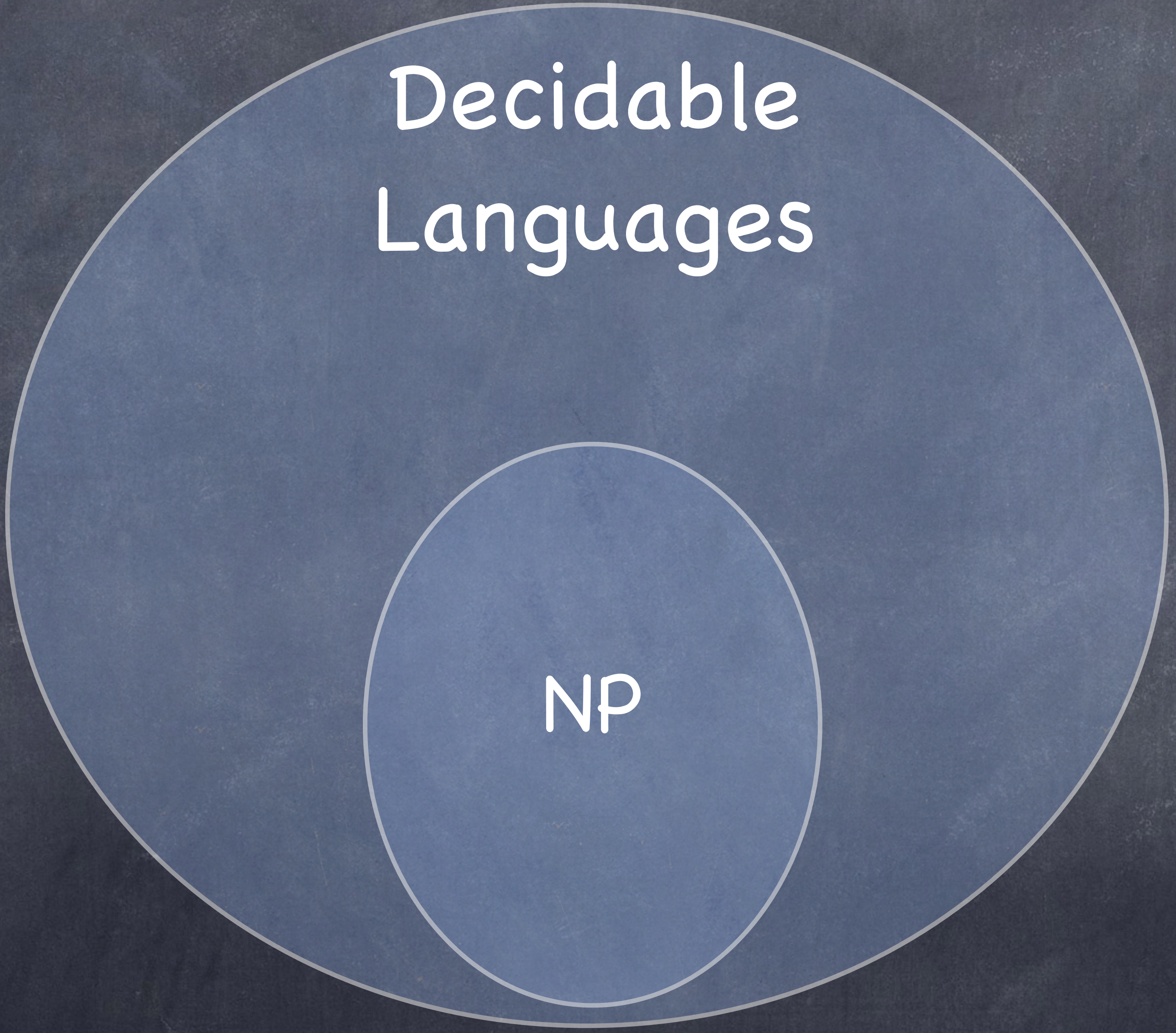
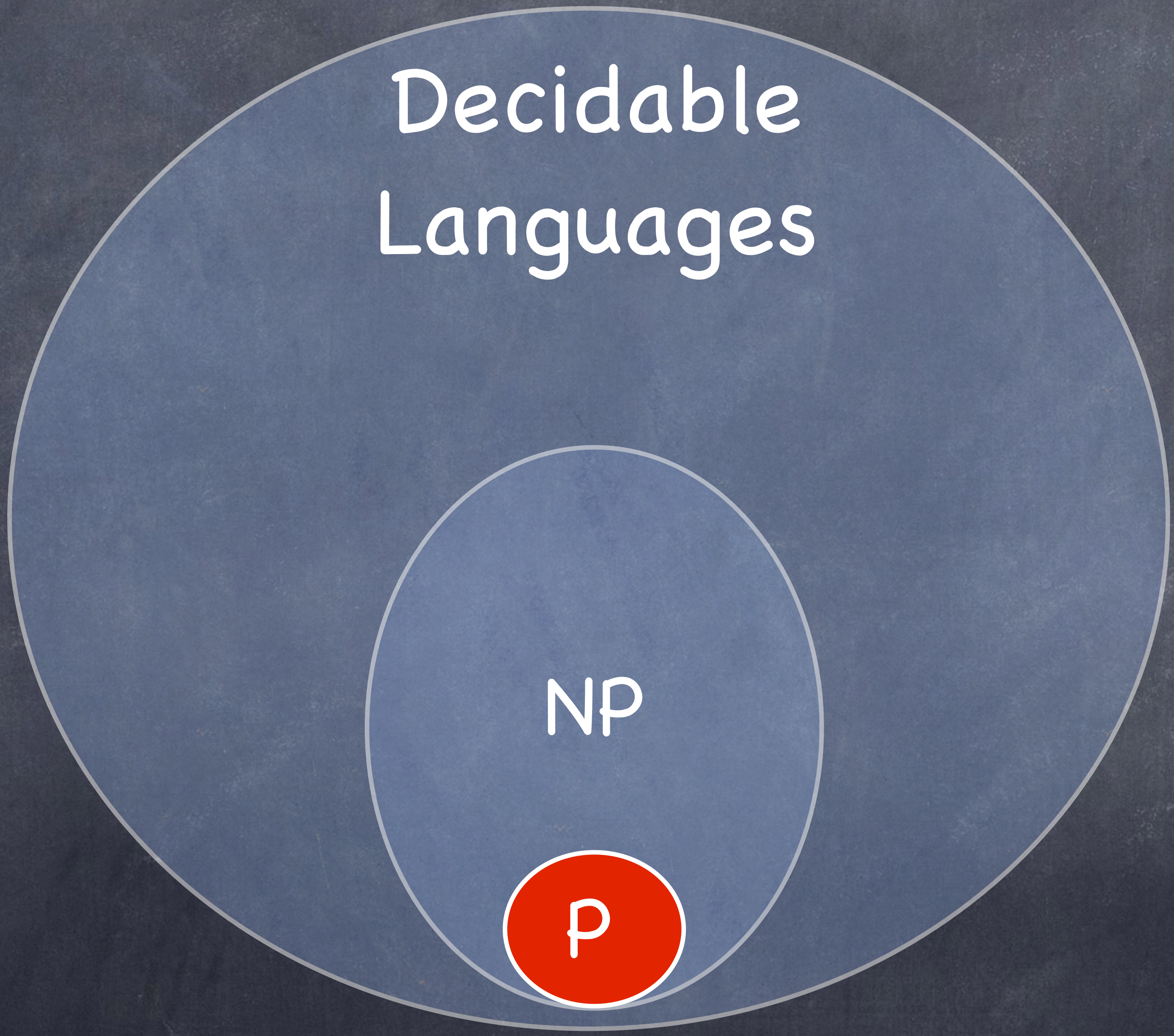
Complexity Theory

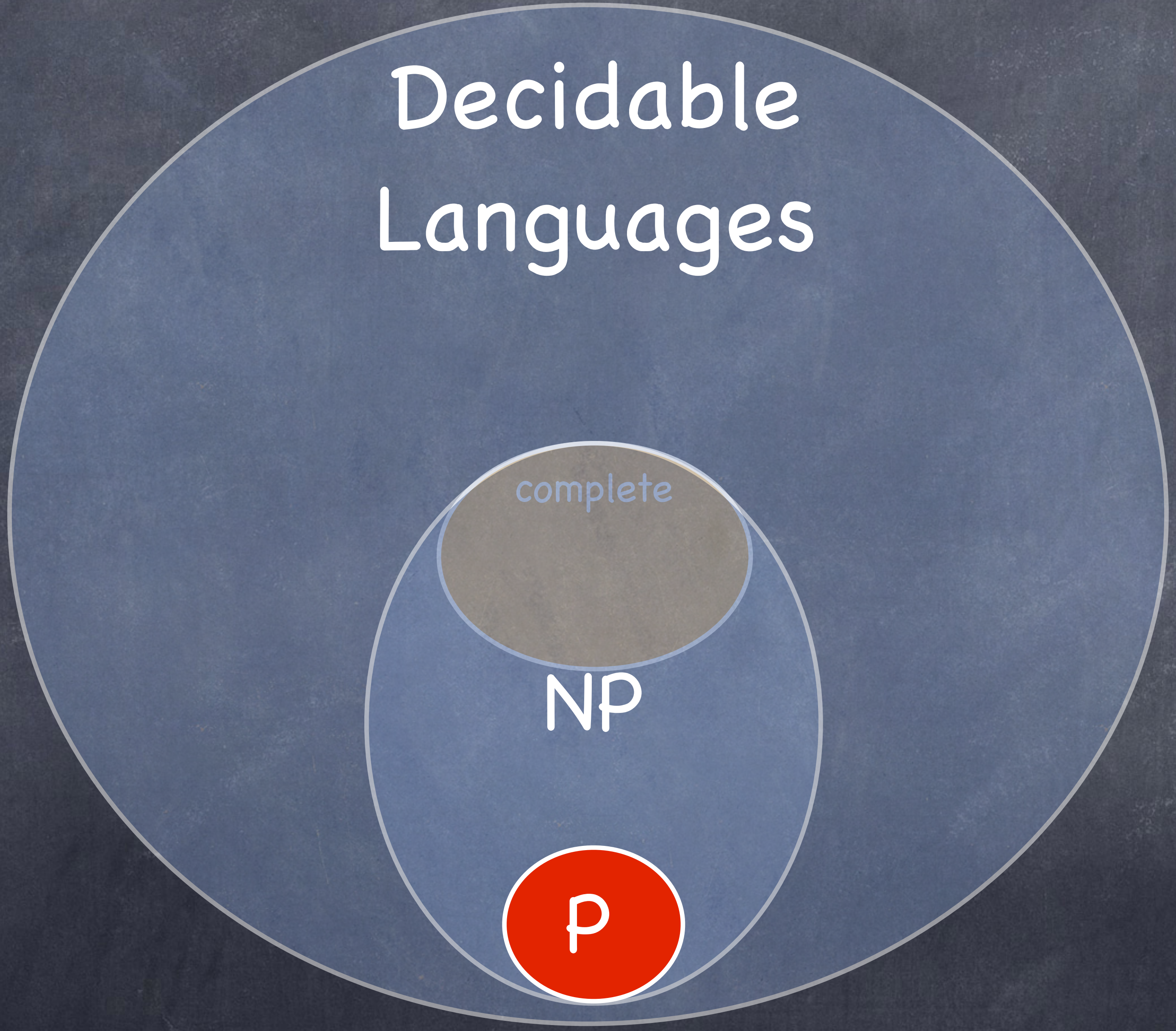
Complexity Theory



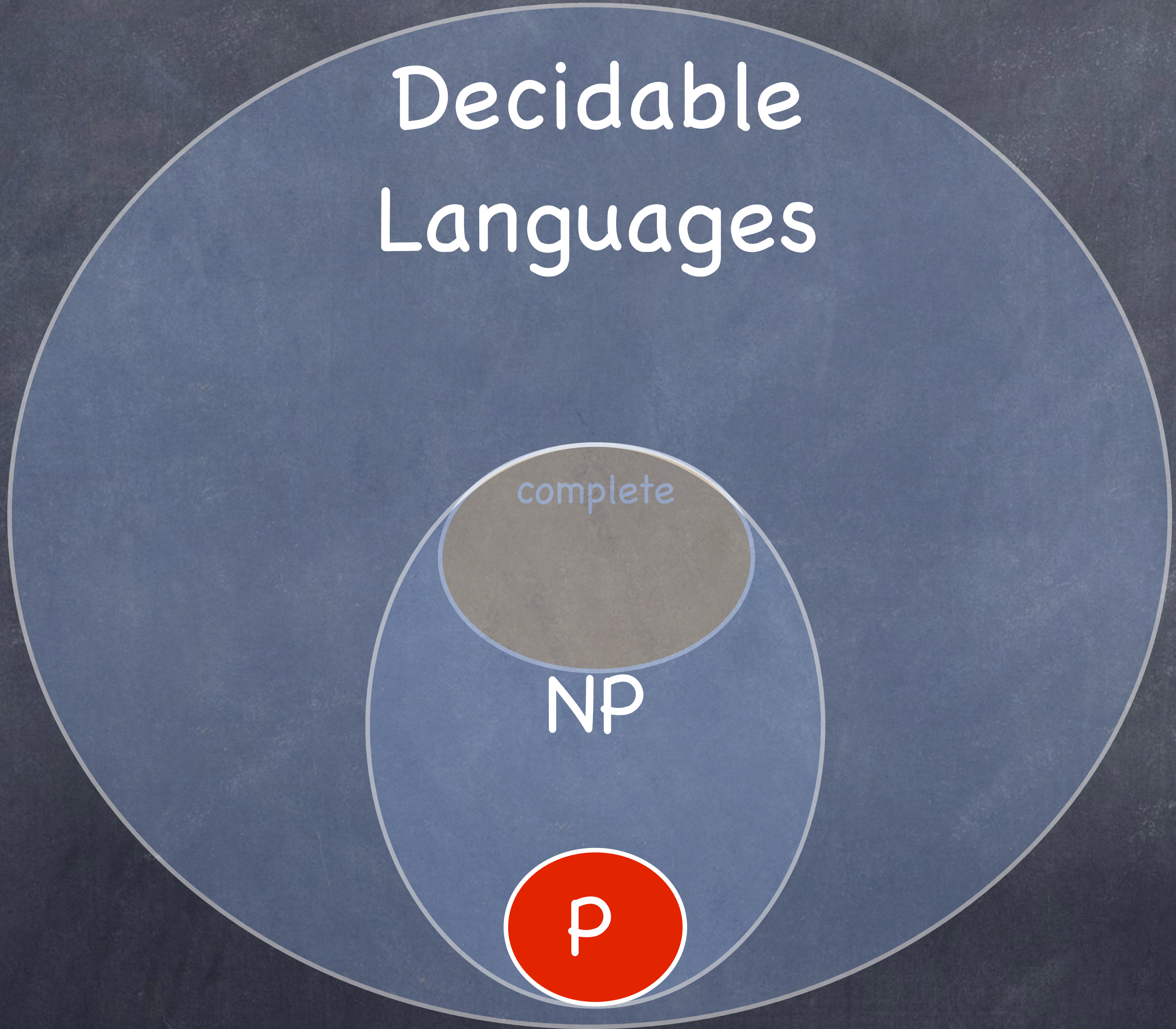
Complexity Theory



Complexity Theory



Complexity Theory



P = NP ?

Beyond NP-Completeness

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- P-Space Completeness: problems that require a reasonable (Poly) amount of **space** to be solved but may use very long time though.

Beyond NP-Completeness

- P-Space Completeness: problems that require a reasonable (Poly) amount of **space** to be solved but may use very long time though.
- Many such problems. If any of them may be solved within reasonable (Poly) amount of time, then all of them can.

P-Space Completeness

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- Geography Game:

  Given a set of country names: Afghanistan, Algeria, Canada, France, Japan, North Korea.

P-Space Completeness

- Geography Game:

  Given a set of country names: Afghanistan, Algeria, Canada, France, Japan, North Korea.

- A two player game: One player chooses a name. The other player must choose a name that starts with the last letter of the previous name and so on. A player wins when his opponent cannot play any name.

Generalized Geography

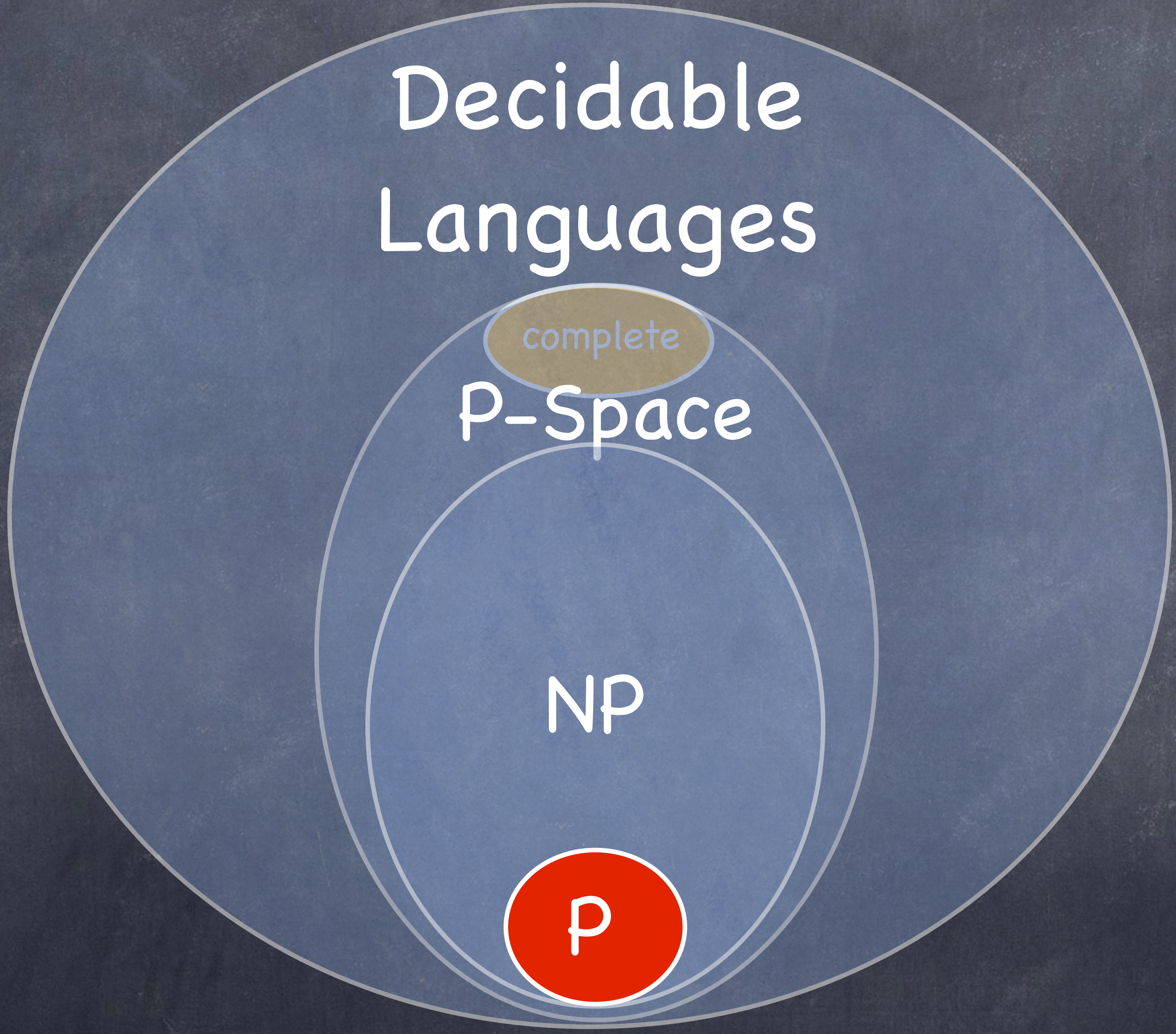
Generalized Geography

- Given an arbitrary set of names: $w_1, \ldots, w_n$.

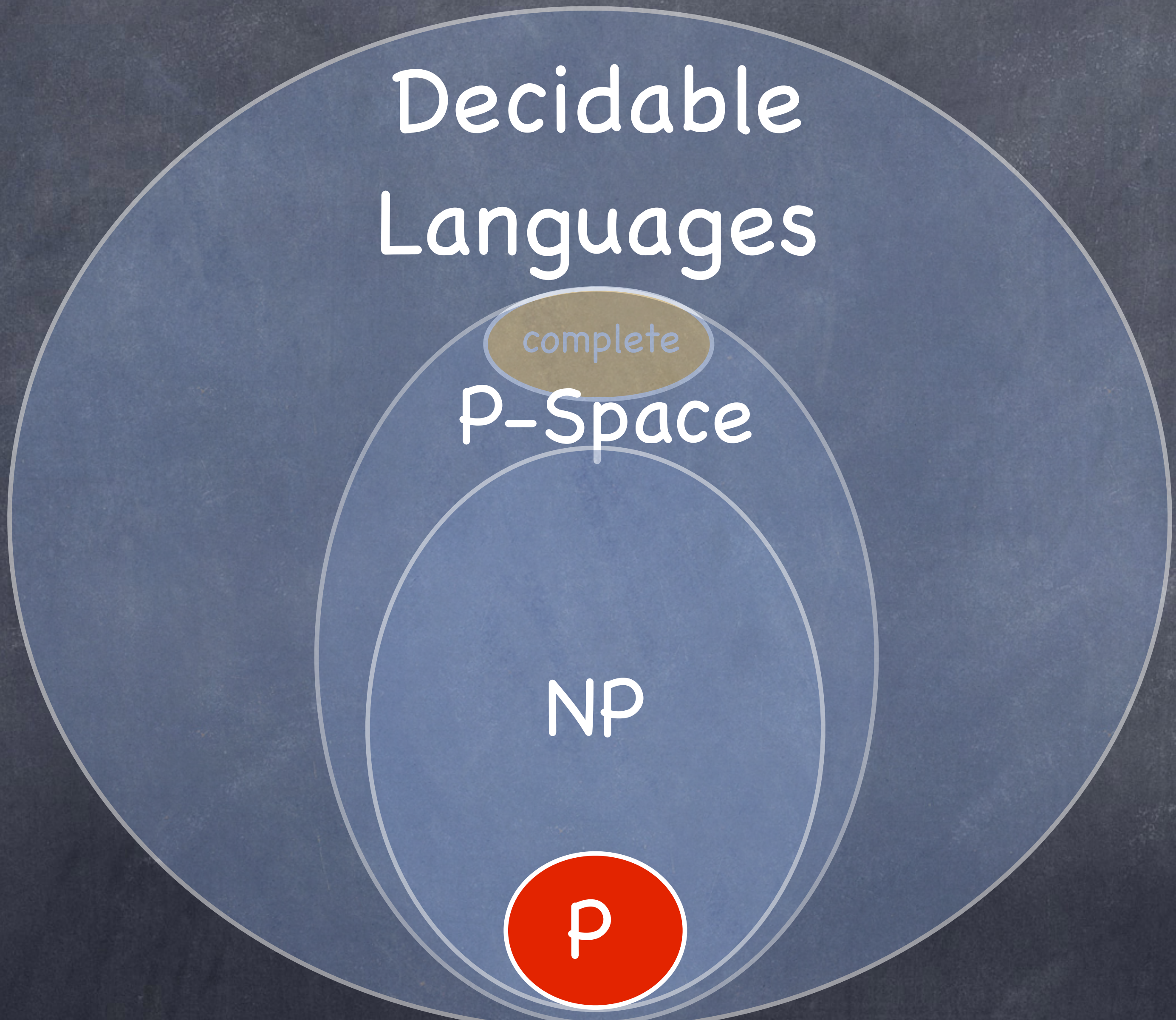
Generalized Geography

- Given an arbitrary set of names: w_1, ..., w_n.
- Is there a winning strategy for the first player to the previous game ?

Complexity Theory



Complexity Theory



NP = P-Space ?

Theoretical Computer Science

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- **Challenges of TCS:**

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- **PROVE** that certain problems are **NOT** computable within a certain time or space. (With applications to cryptography)

Theoretical Computer Science

- <u>**Challenges of TCS:**</u>
- **FIND** efficient solutions to many problems.
- **PROVE** that certain problems are **NOT** computable within a certain time or space. (With applications to cryptography)
- Consider new models of computation. (Such as a Quantum Computer)

Afghanistan

Afghanistan 2

Albania

Albania 2

Albania 3

Albania 4

Algeria

Andorra

Andorra 2

Angola

Angola 2

Antigua and Barbuda

Antigua and Barbuda 2

Argentina

Armenia

Armenia 2

Australia

Australia 2

Australia 3

Austria

Austria 2

Azerbaijan

Azerbaijan 2

Bahamas, The

PCP with constraints

- input $(a^{n_1}/a^{m_1}), (a^{n_2}/a^{m_2})$
- find $k_1,k_2 \geq 0$ s.t. $k_1n_1+k_2n_2=k_1m_1+k_2m_2$
- find $k_1,k_2 \geq 0$ s.t $k_1(n_1-m_1)=k_2(m_2-n_2)$
- if $n_1=m_1$ then set $k_1=1$, $k_2=0$
- else if $n_2=m_2$ then set $k_1=0$, $k_2=1$
- else if $(n_1-m_1)(n_2-m_2)<0$ then set $k_1=|n_2-m_2|,k_2=|n_1-m_1|$
- else no solution exists