COMP-330
Theory of Computation

Fall 2019 -- Prof. Claude Crépeau

Lec. 8 : Regular and NON-Reg. Languages

GNFA → Reg. Expression

CLAIM 1.65

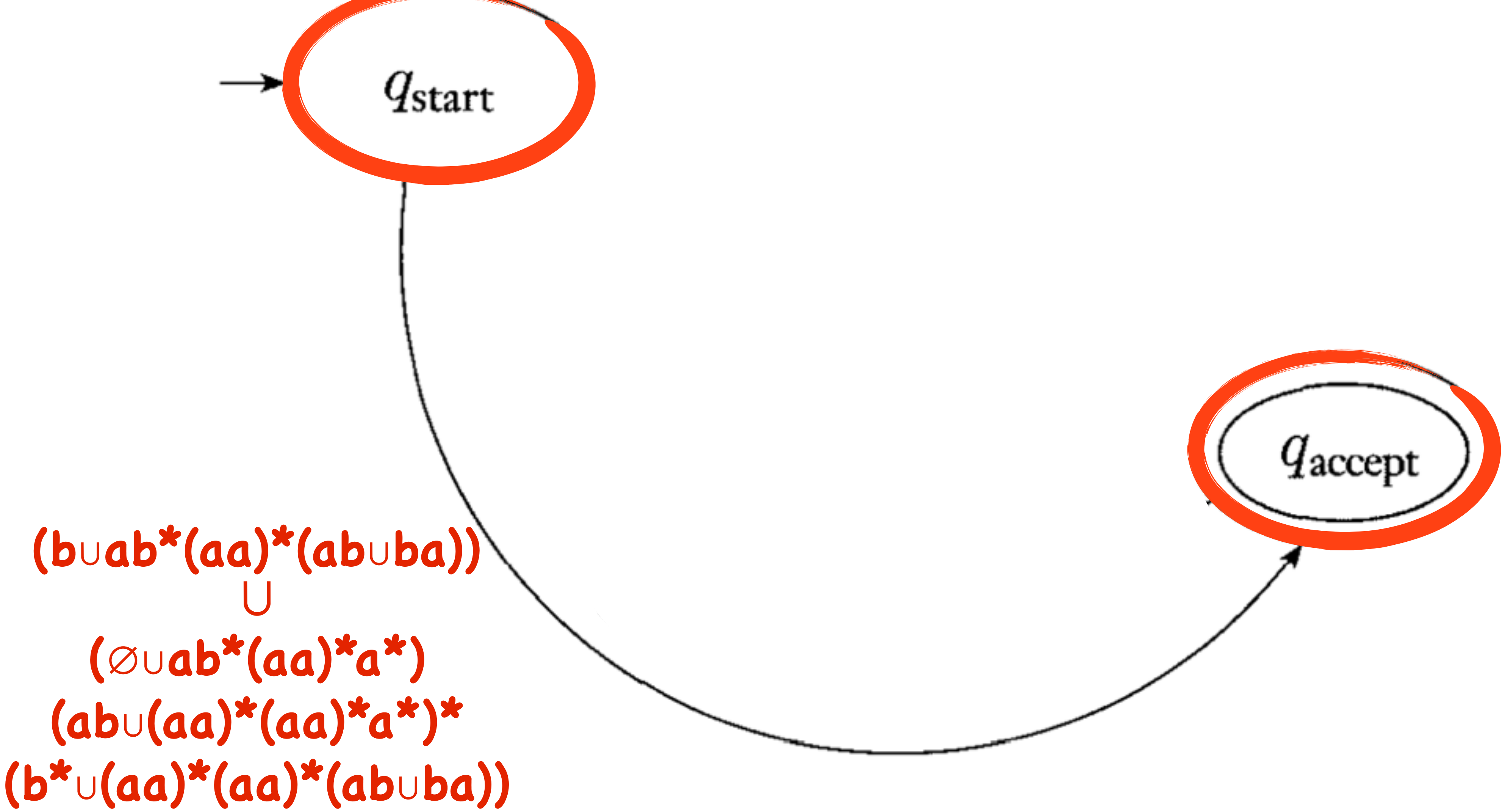
For any GNFA G, CONVERT(G) is equivalent to G.

We prove this claim by induction on k, the number of states of the GNFA.

"equivalent" means L(CONVERT(G)) = L(G)

GNFA → Reg. Expression

- Induction basis
- Let G be a GNFA with exactly k=2 states.
- Because of the special form of our GNFA, the two states are the start and accept states. The regular expression on the transition from q_{start} to q_{accept} generates the language accepted by this GNFA.



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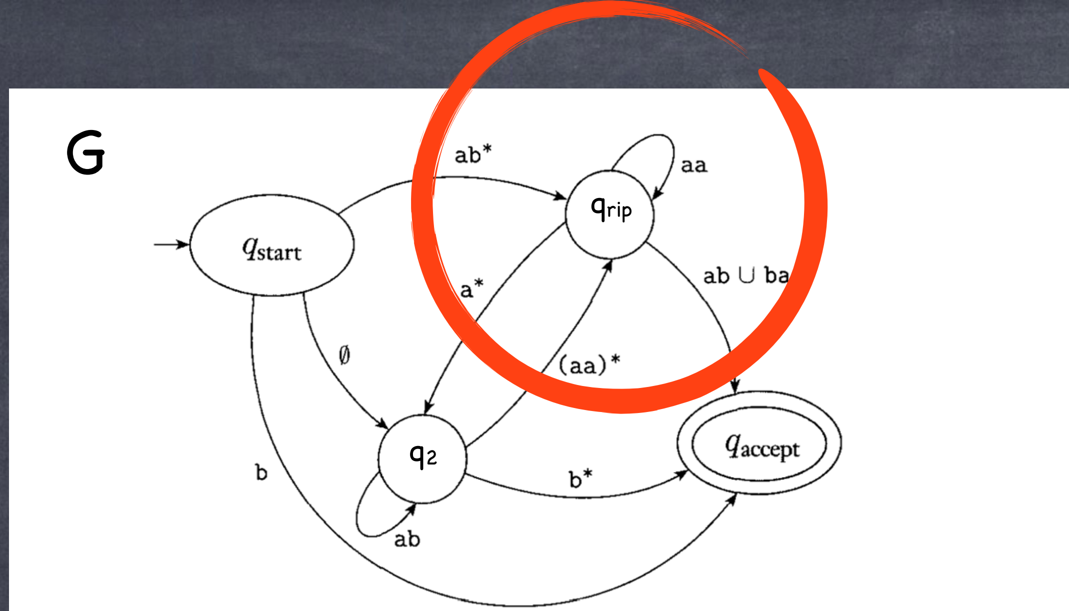

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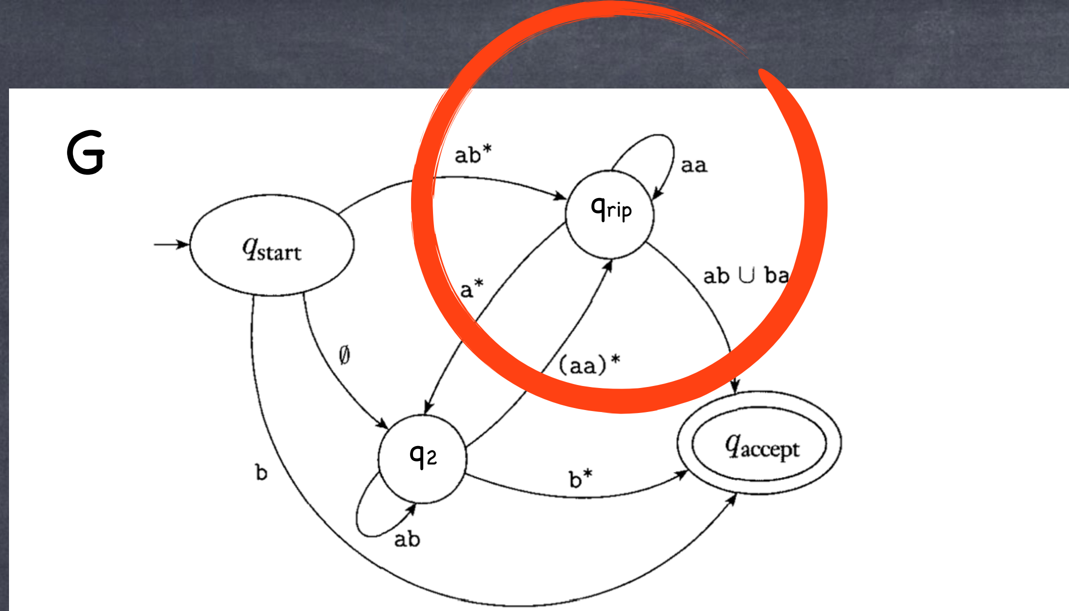

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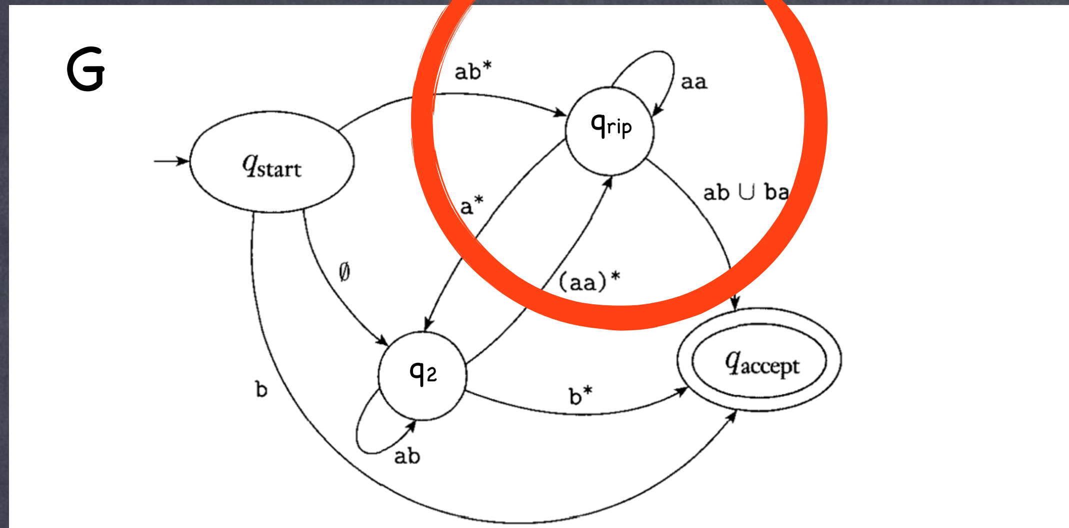

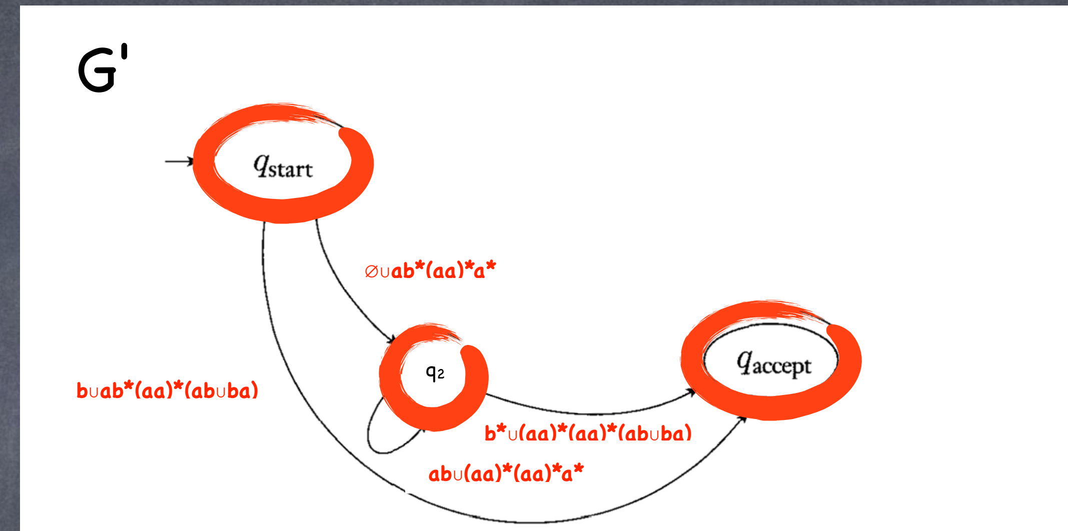

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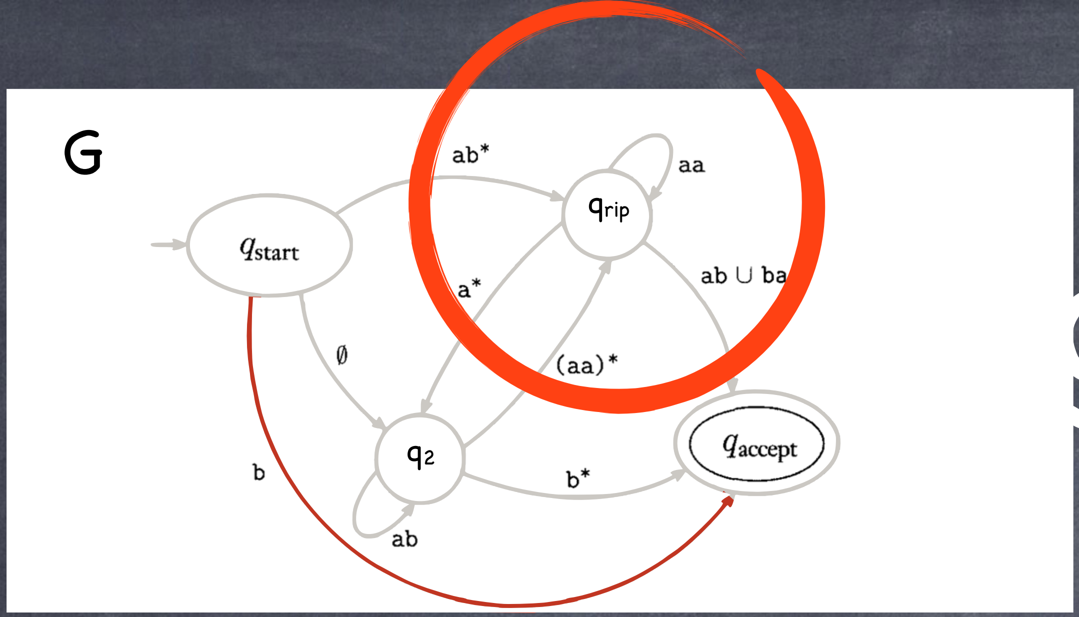

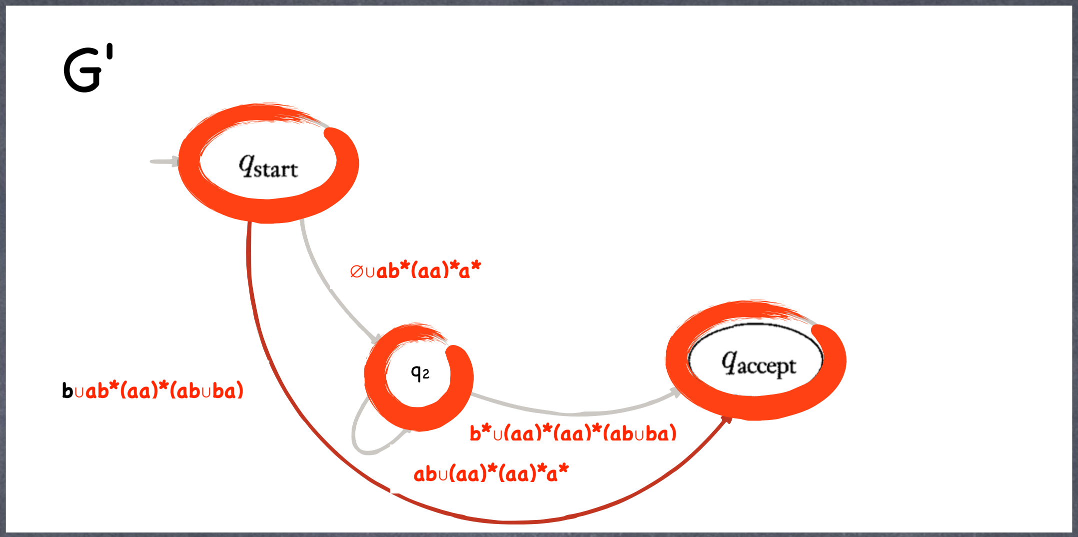

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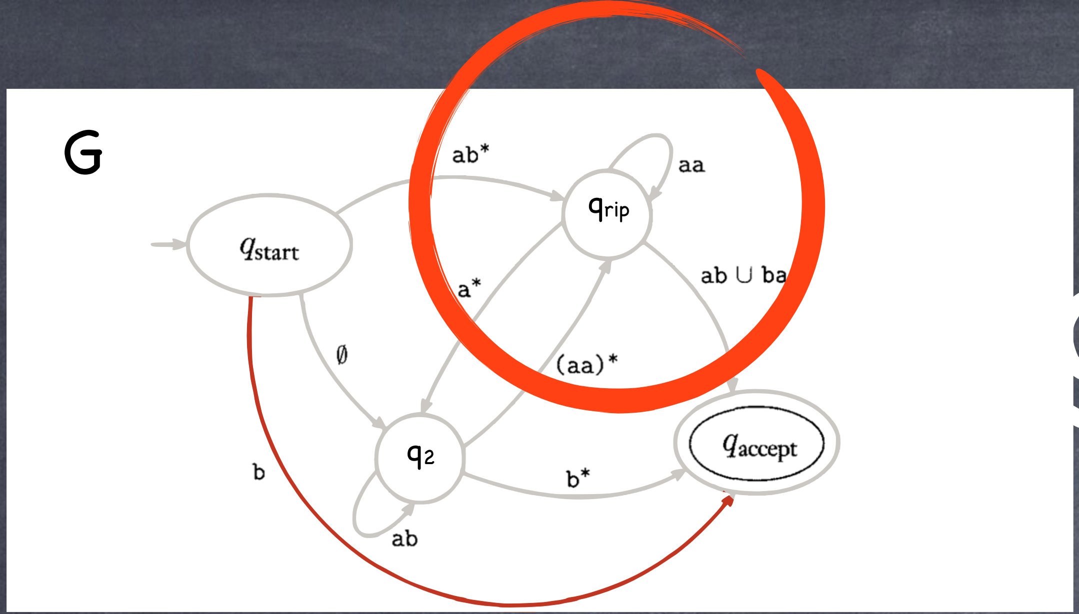

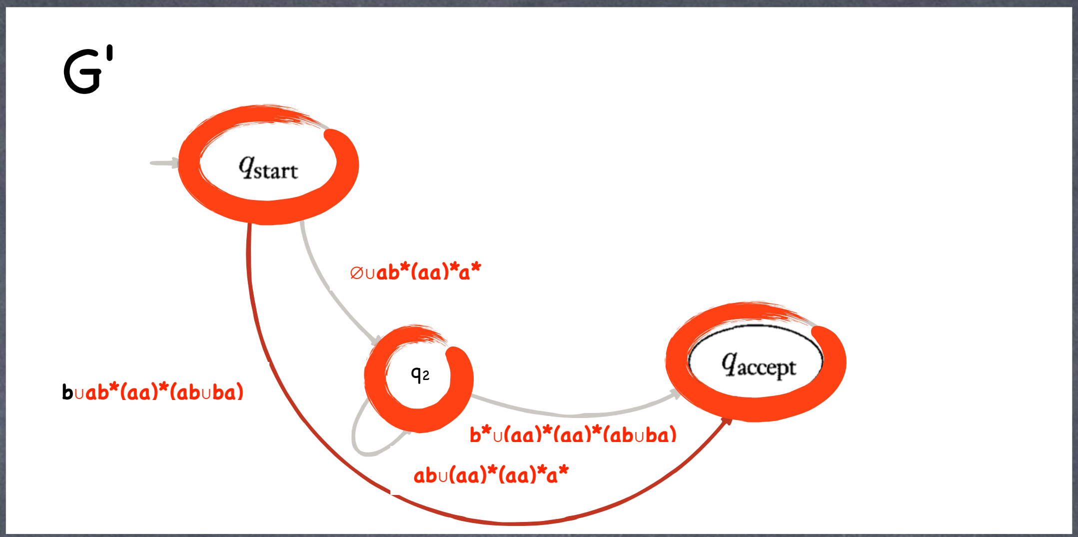

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- If q_{rip} is not a state of the sequence, then the very same exact sequence will accept w in G′ because its transitions R_4 contain all those R_4 in G (except for q_{rip}) in a union with new possibilities related to ripping q_{rip}.

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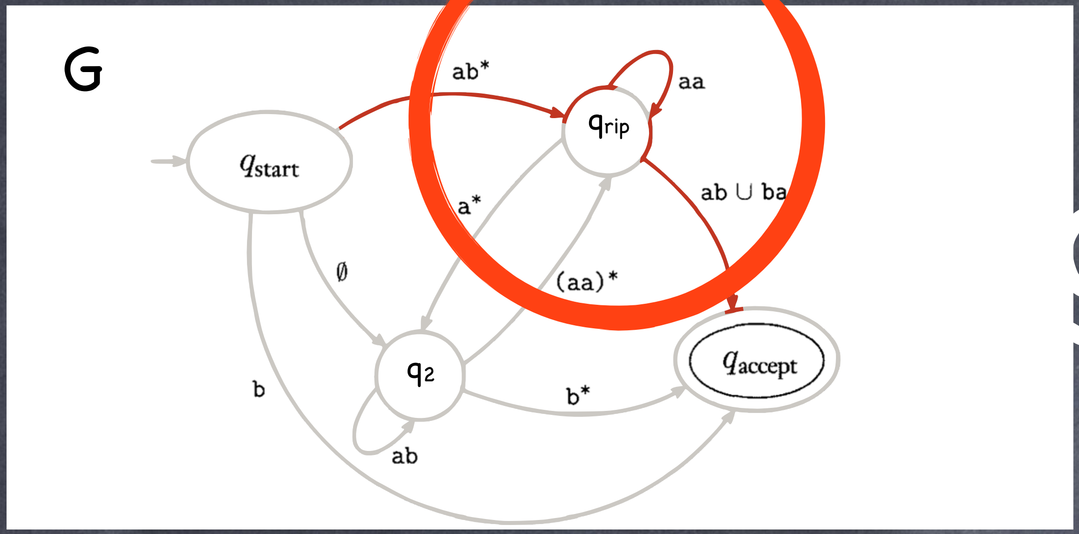
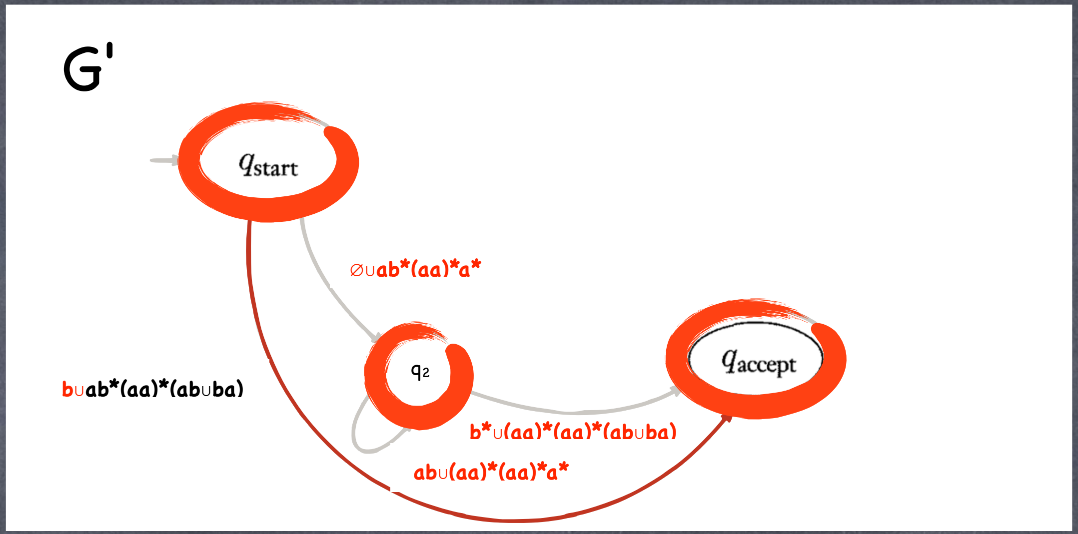
- If q_{rip} is a state of the sequence, then the same sequence (but with all q_{rip} removed) will accept w in G′. That's because any three elements in a row q_i,q_{rip},q_j ($q_i \neq q_{rip} \neq q_j$) in G's accepting sequence, will be processed identically through states q_i,q_j in G′. Remember that the transitions for q_i,q_j in G′ contain all those $R_1(R_2)^*R_3$ from G involving q_{rip} in a union with older possibilities (R_4). (we can deal with $q_i,q_{rip},\ldots,q_{rip},q_j$ similarly.)





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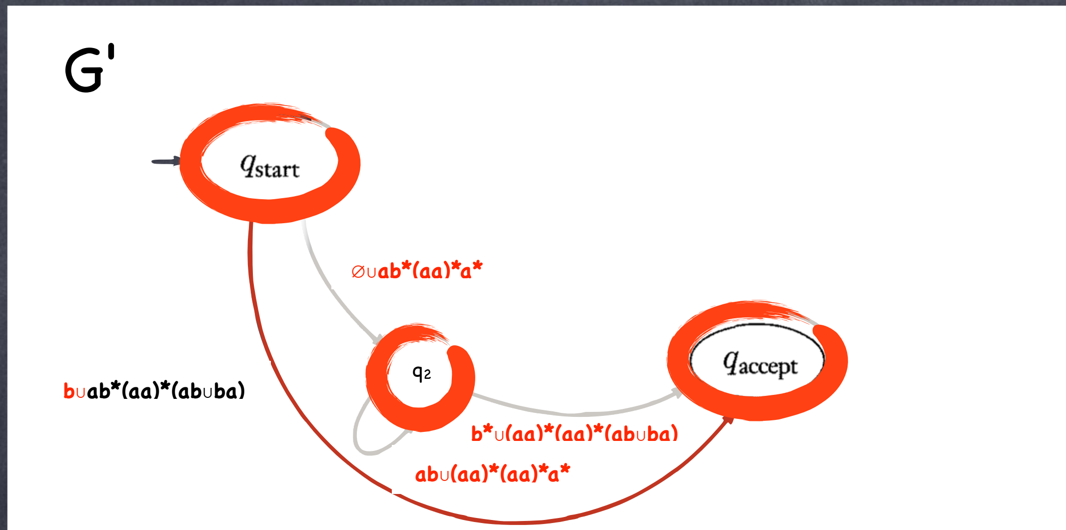
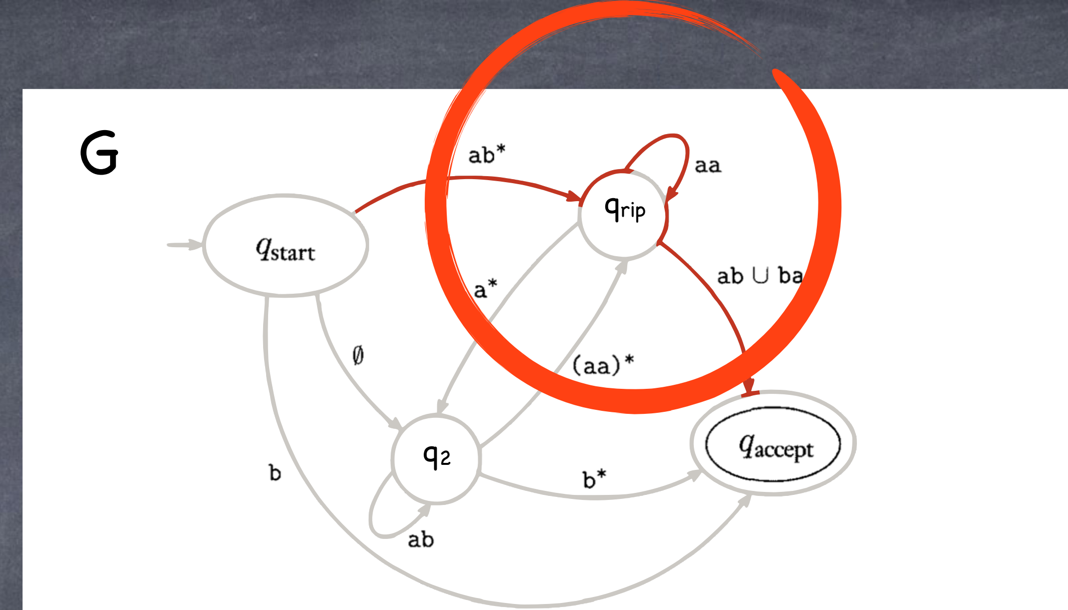
- This proved "if $w \in L(G)$ then $w \in L(G')$". We should also prove "if $w \in L(G')$ then $w \in L(G)$".
- Let w be a string accepted by G', i.e. $w \in L(G')$. Consider an accepting sequence $q_{start}, q_1, q_2, \ldots, q_{accept}$ for string w. Consider any two consecutive states q_i, q_{i+1}. The same portion of w is processed in G in either part of the union, $R_1(R_2)^*R_3$ or R_4, along the transition between q_i and q_{i+1}.

eg.





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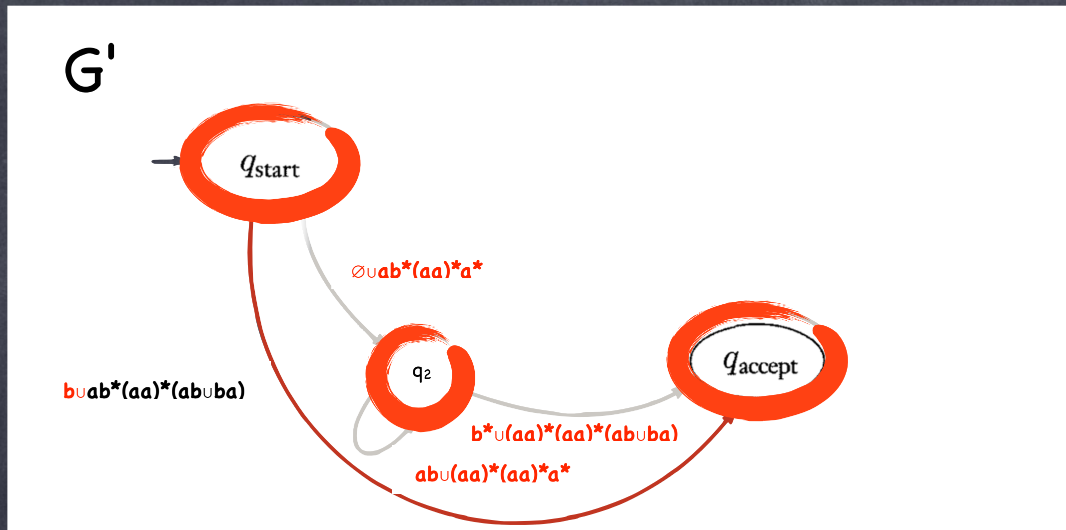
- We conclude that if $w \in L(G')$ then $w \in L(G)$.



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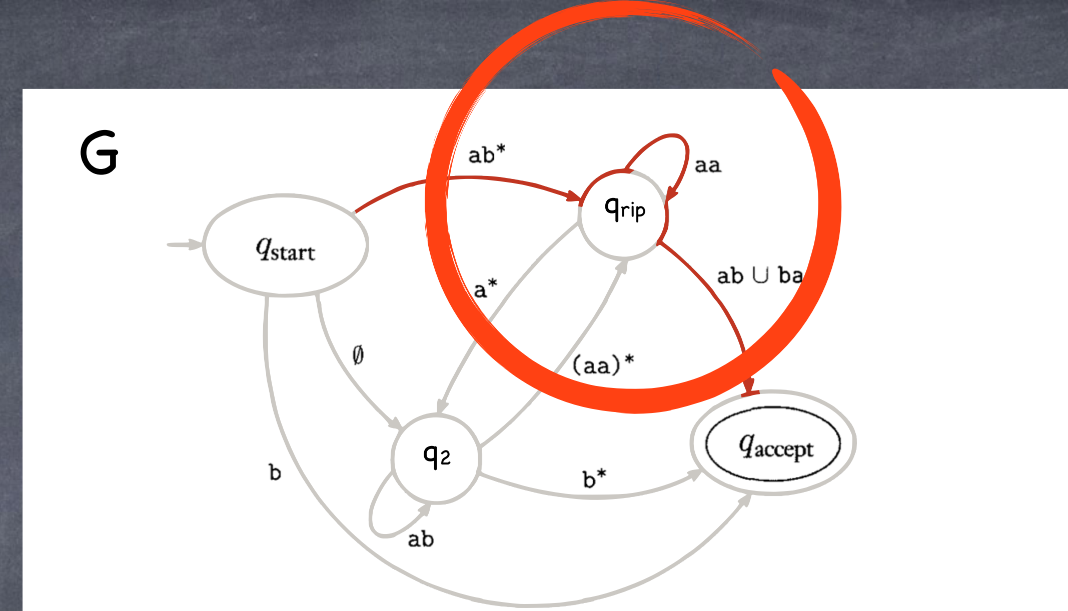

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- Combining both statements we get L(G′)=L(G).
- By induction hypothesis L(G′)=L(CONVERT(G′)) because G′ contains k-1 states. By construction, CONVERT(G)=CONVERT(G′). Therefore

  L(G)=L(CONVERT(G))=L(CONVERT(G′))=L(G′).

QED

DFA → GNFA → Reg. Exp.

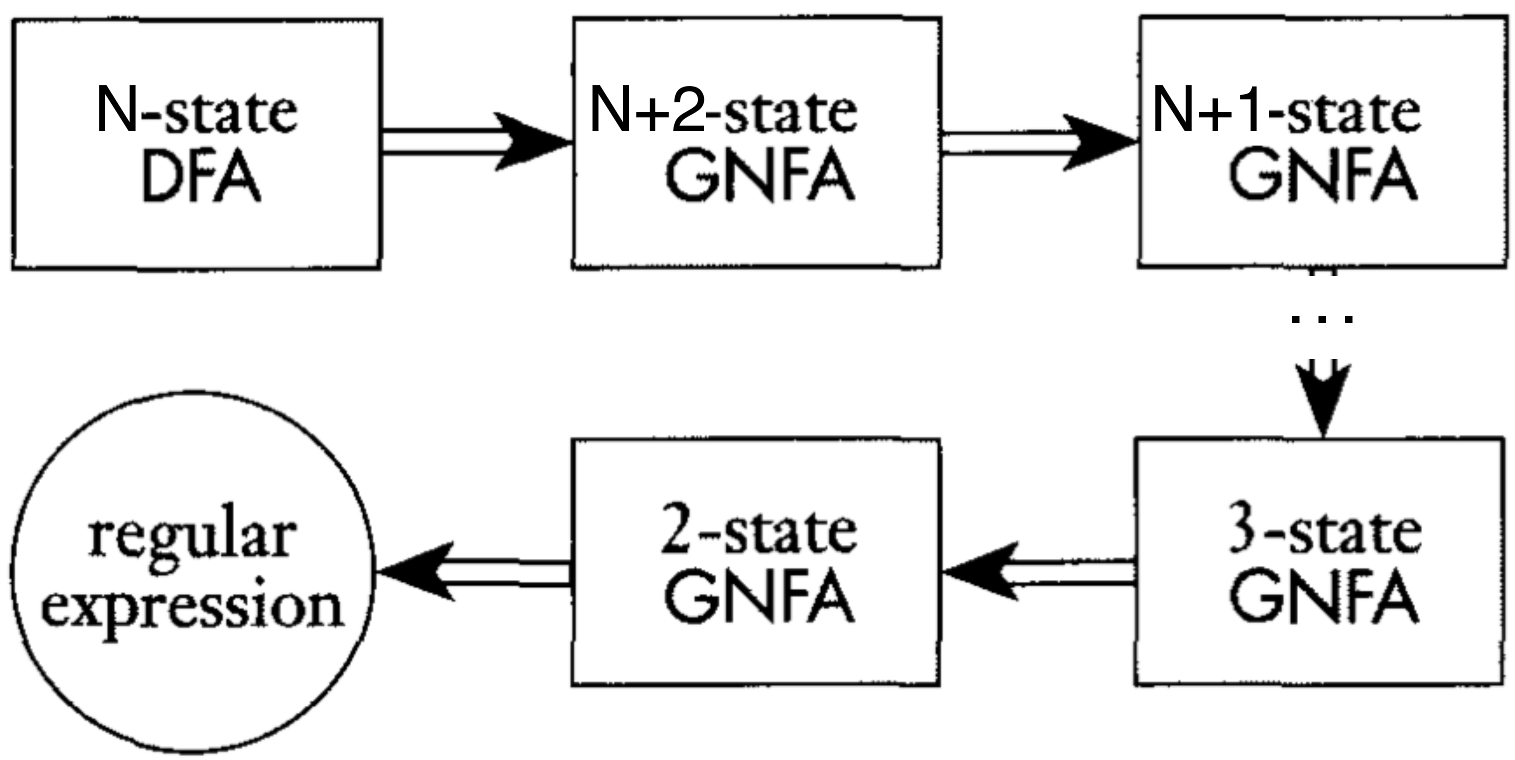
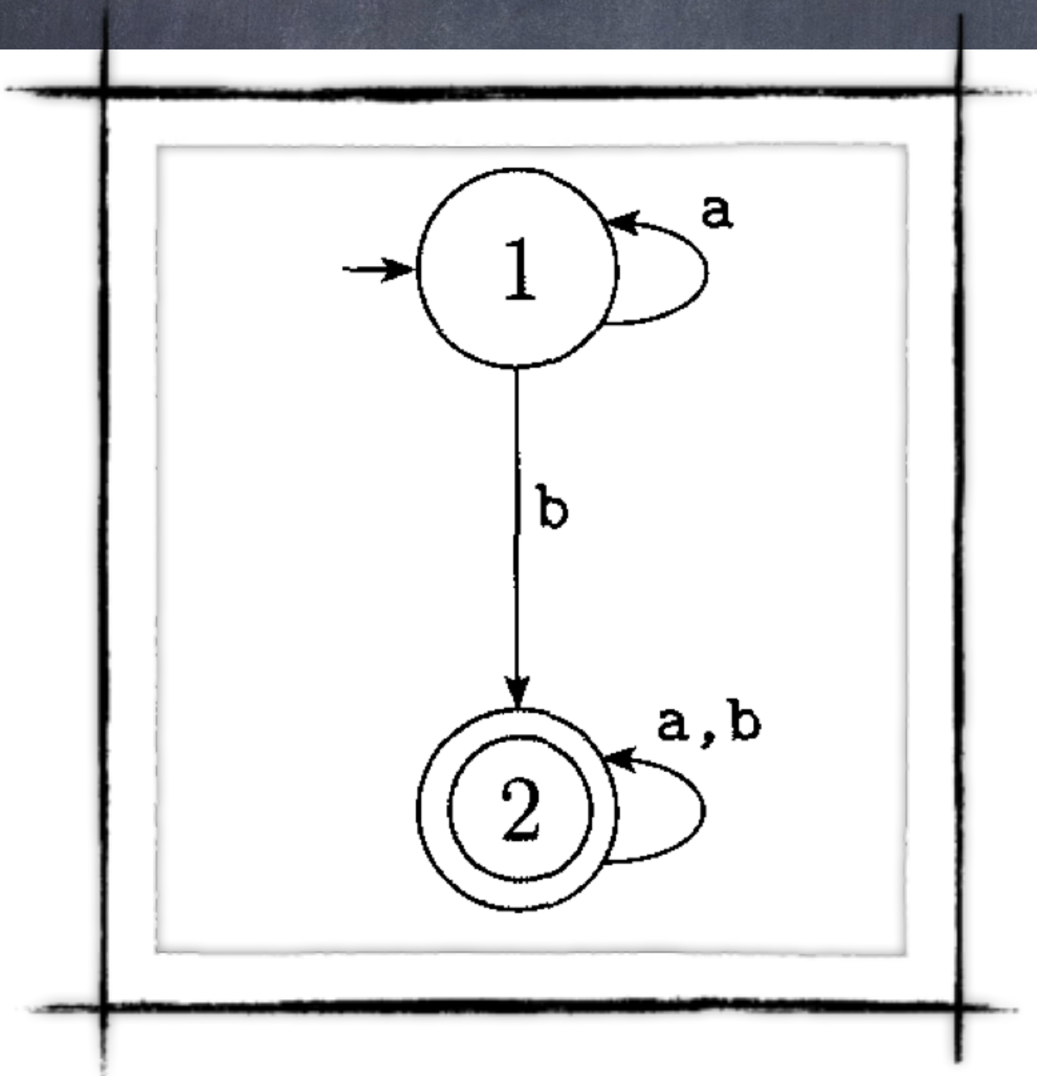
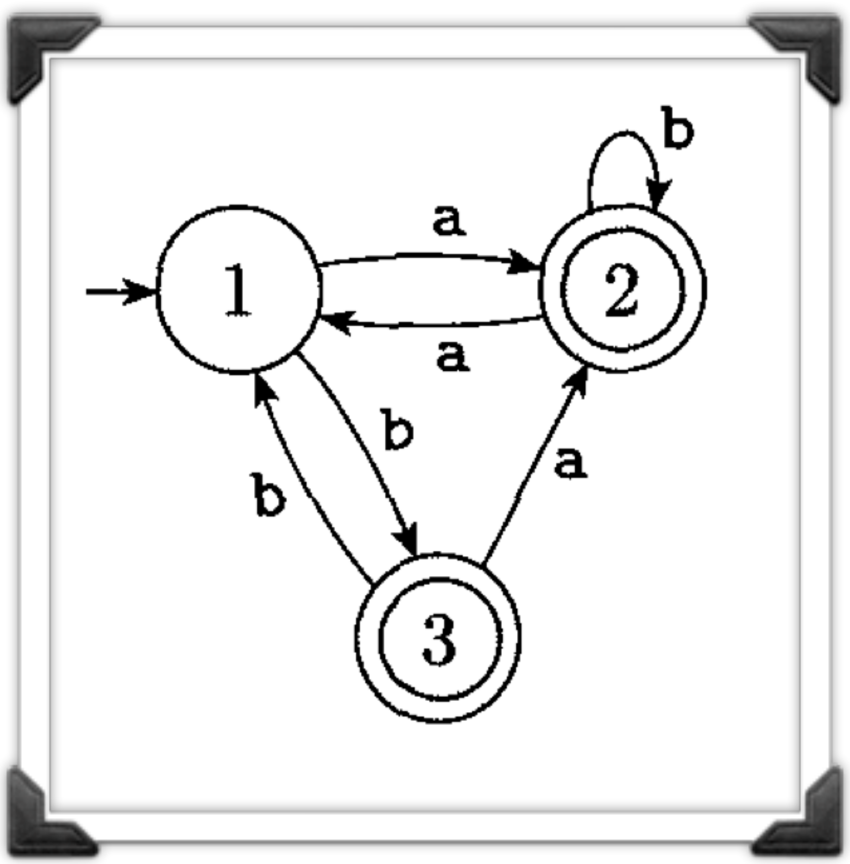


FIGURE 1.62
Typical stages in converting a DFA to a regular expression

DFA → GNFA → Reg. Exp.

- Two examples





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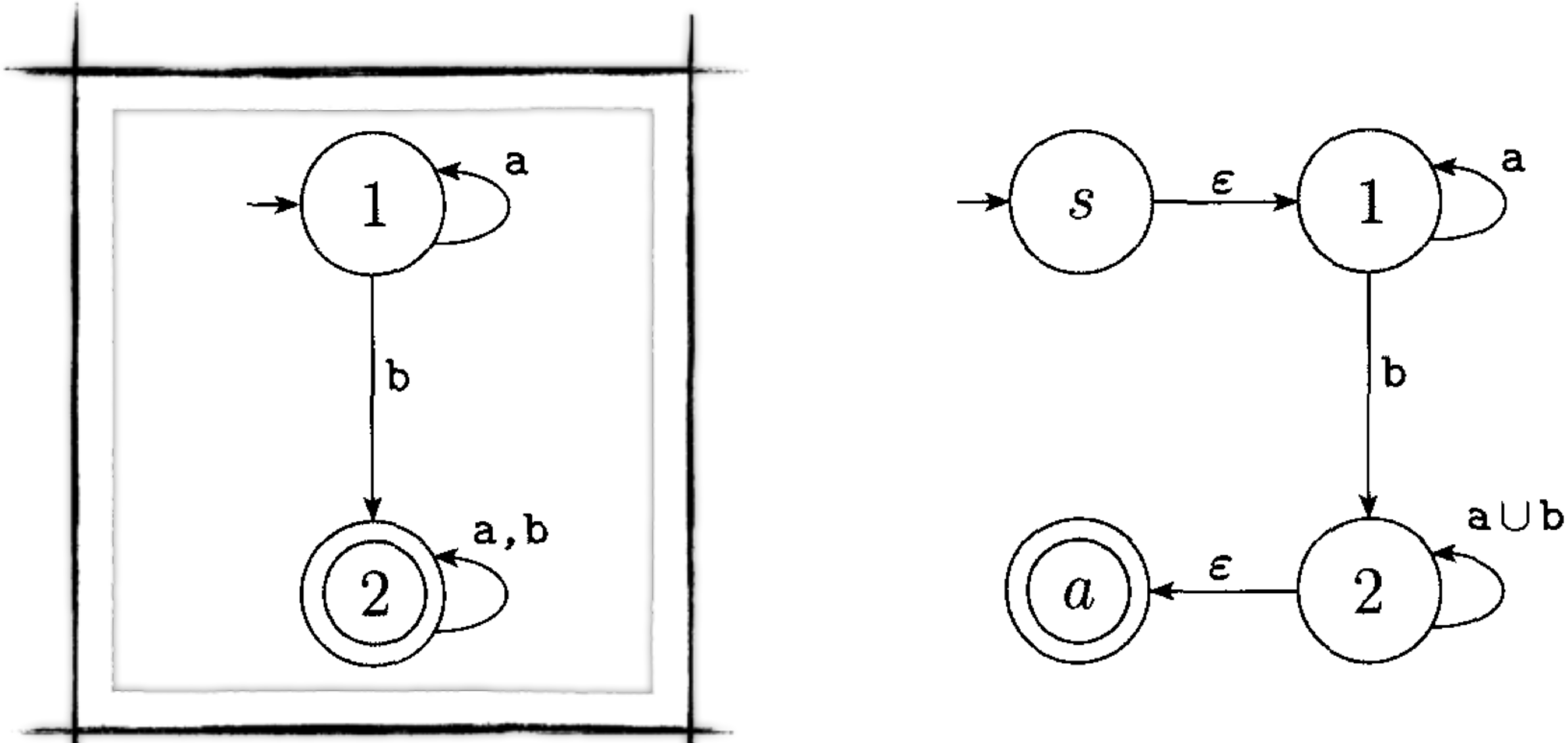

(a) (b)

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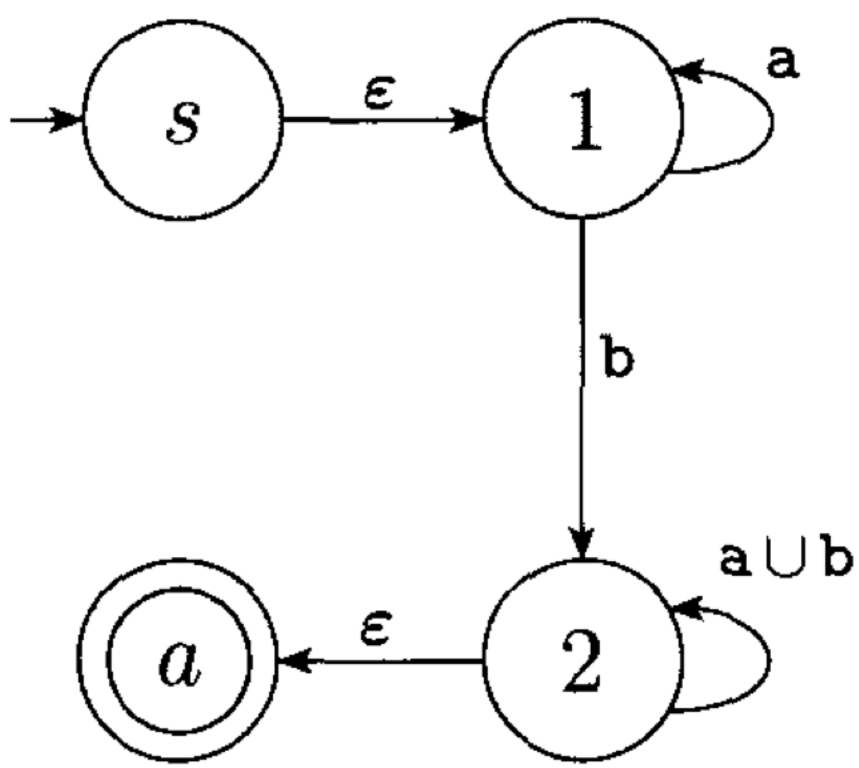
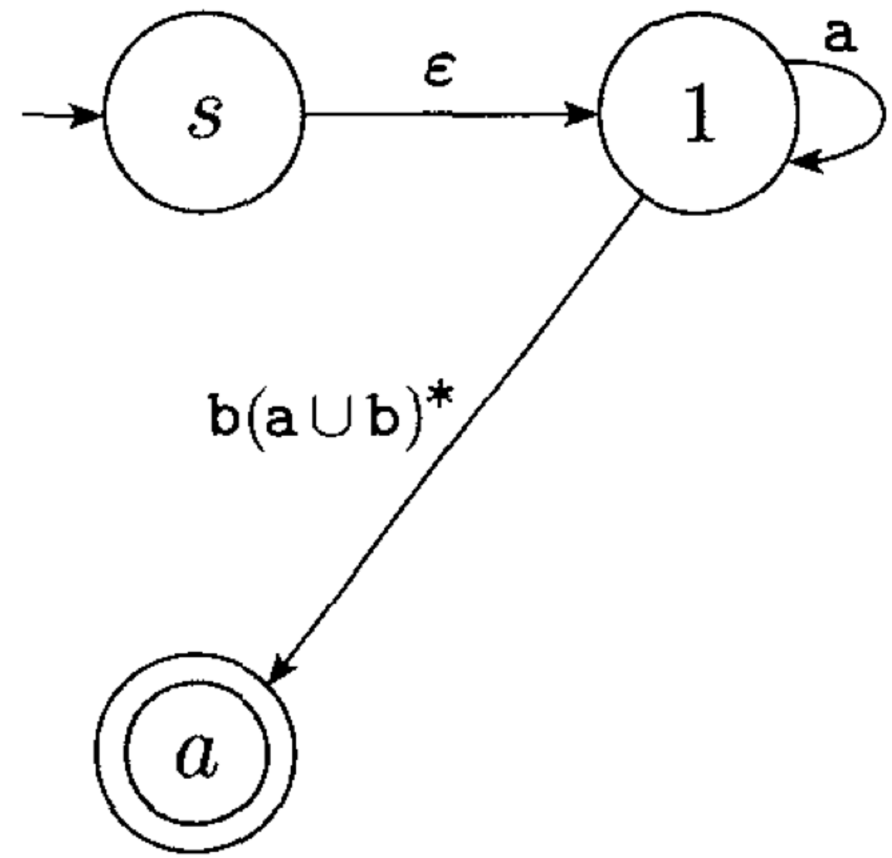
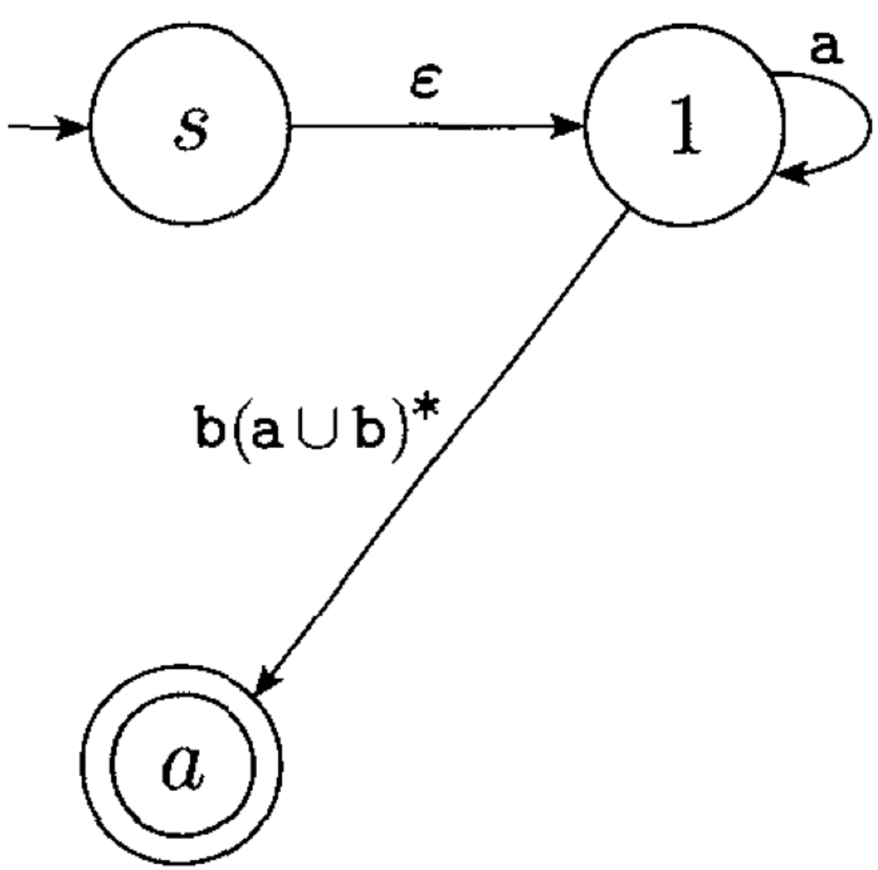


(b)



(c)

DFA → GNFA → Reg. Exp.



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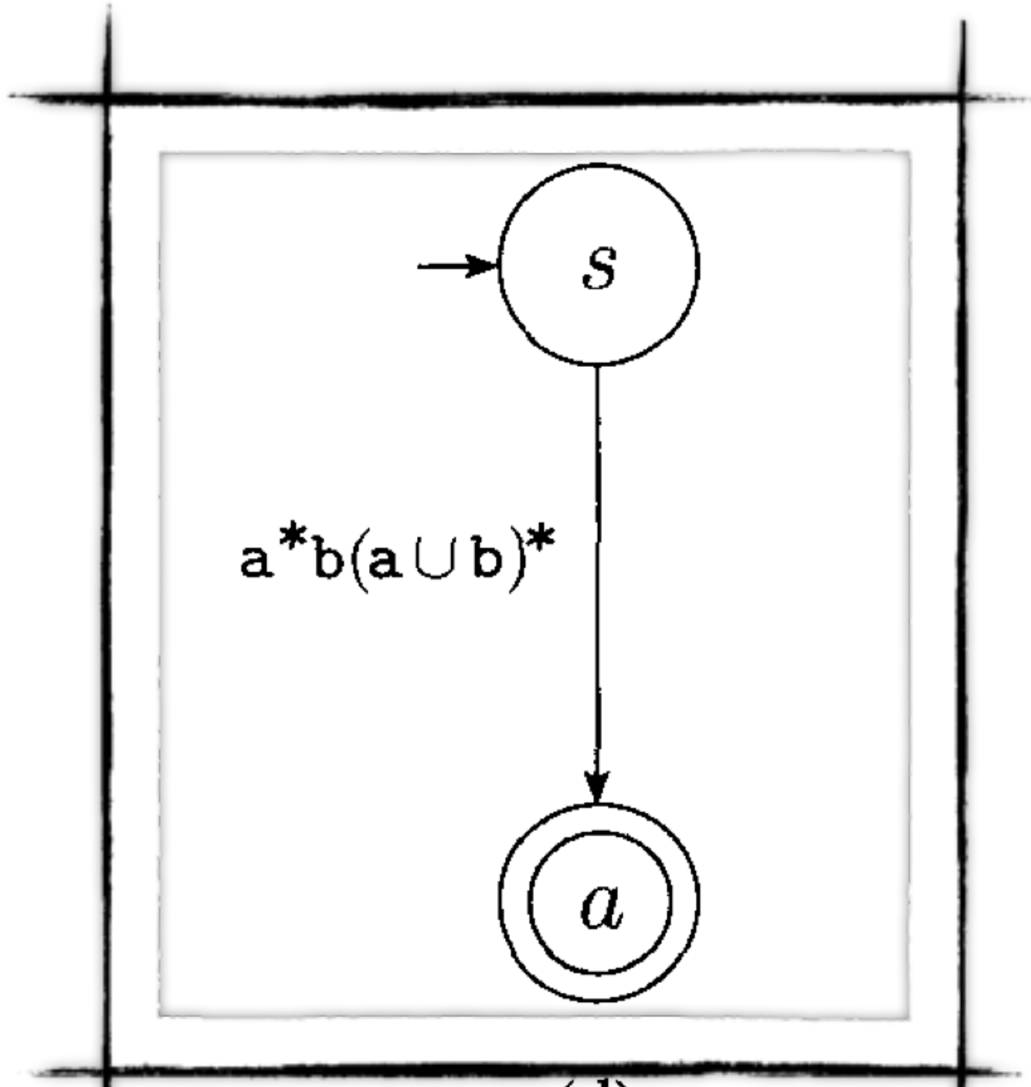

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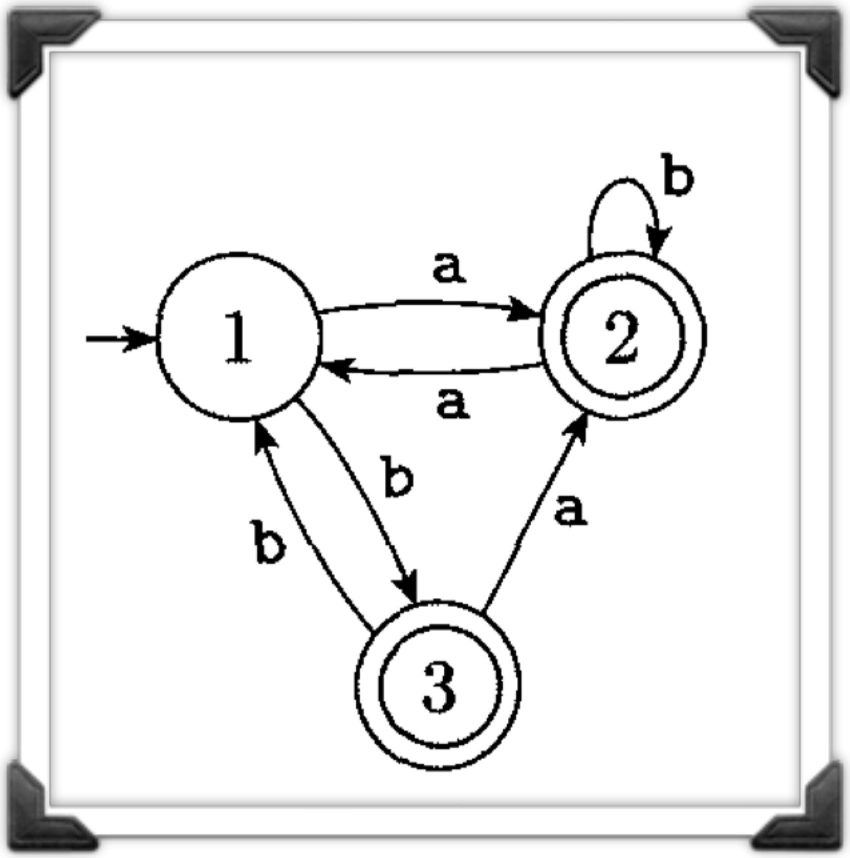

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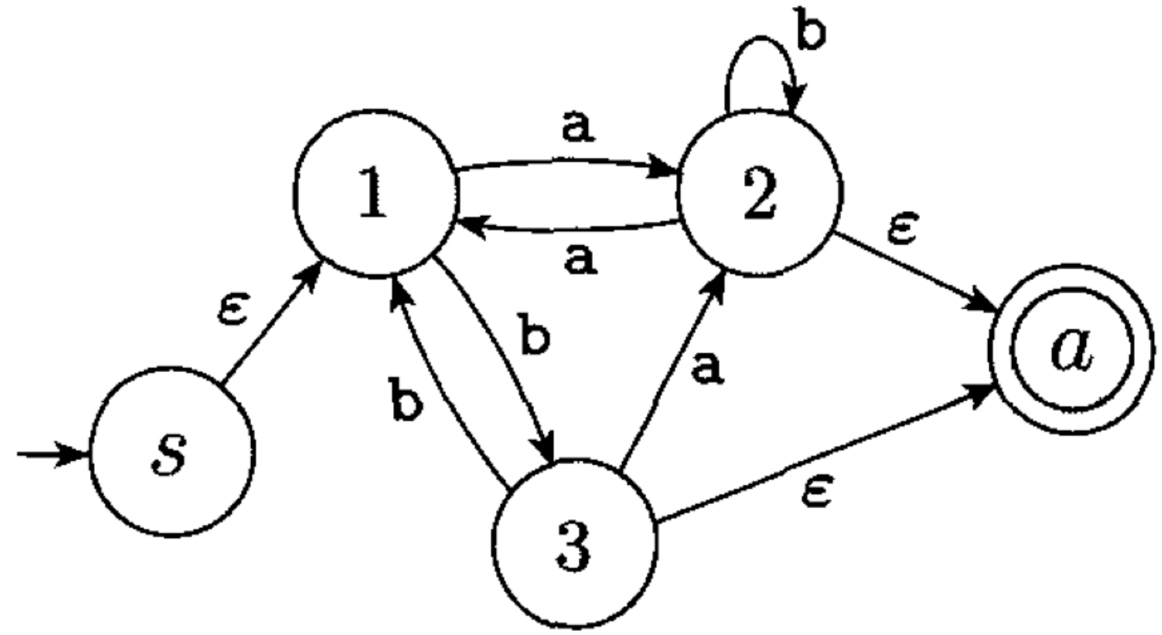

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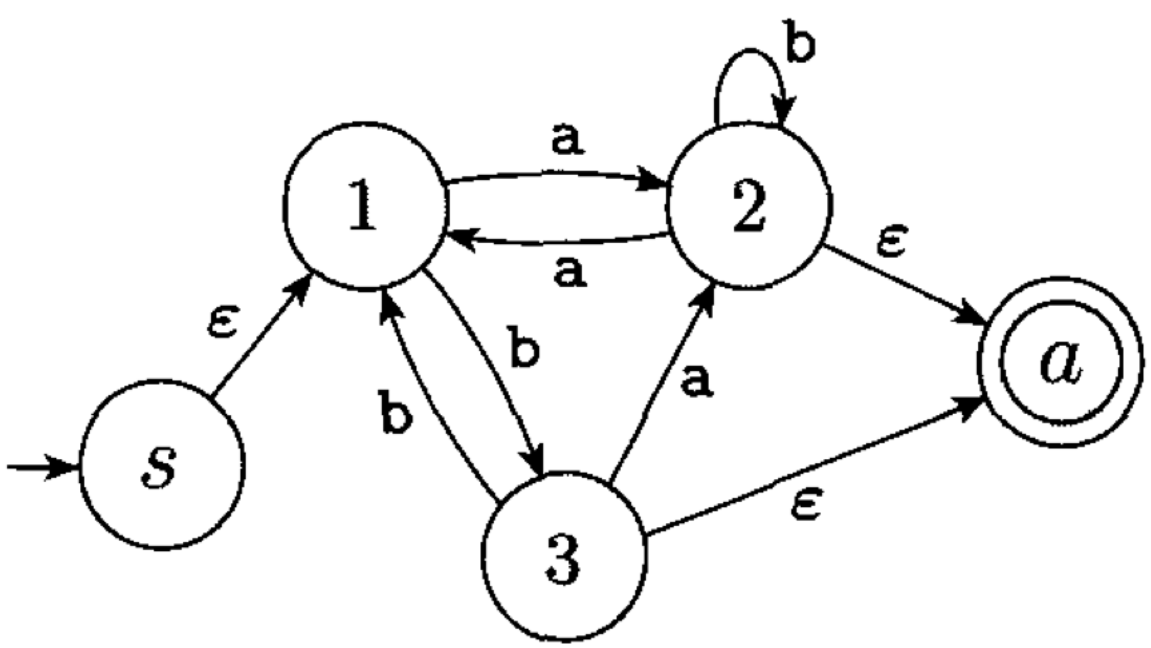

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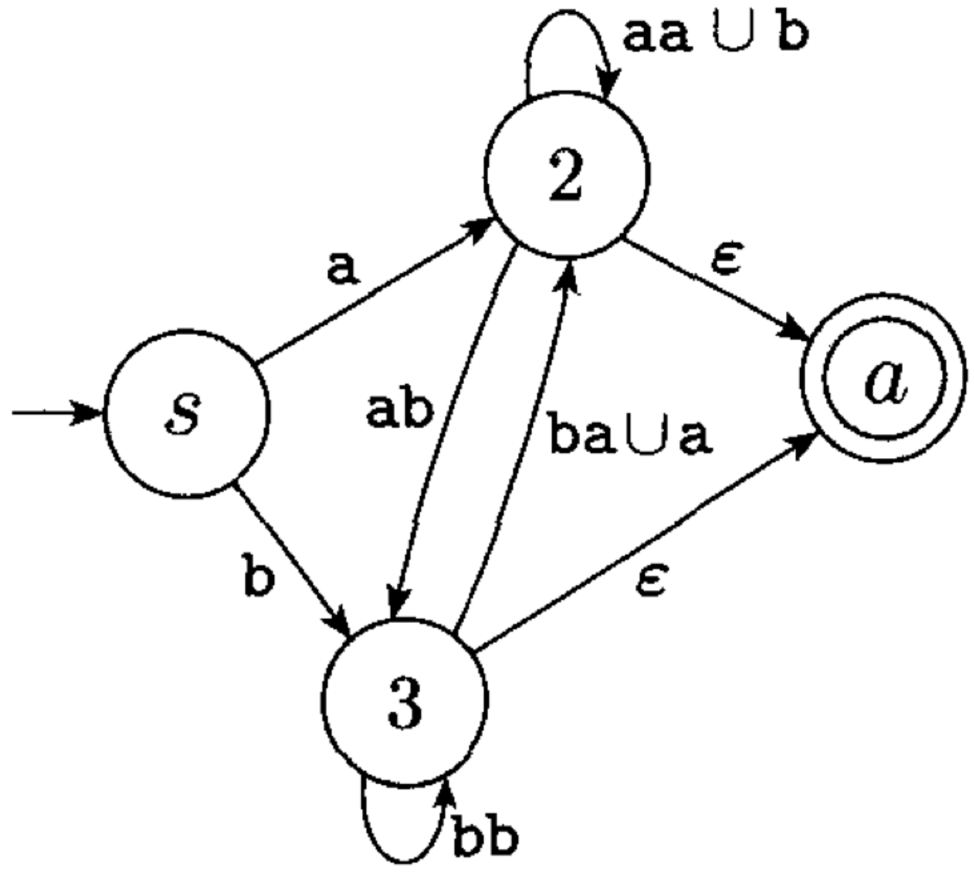

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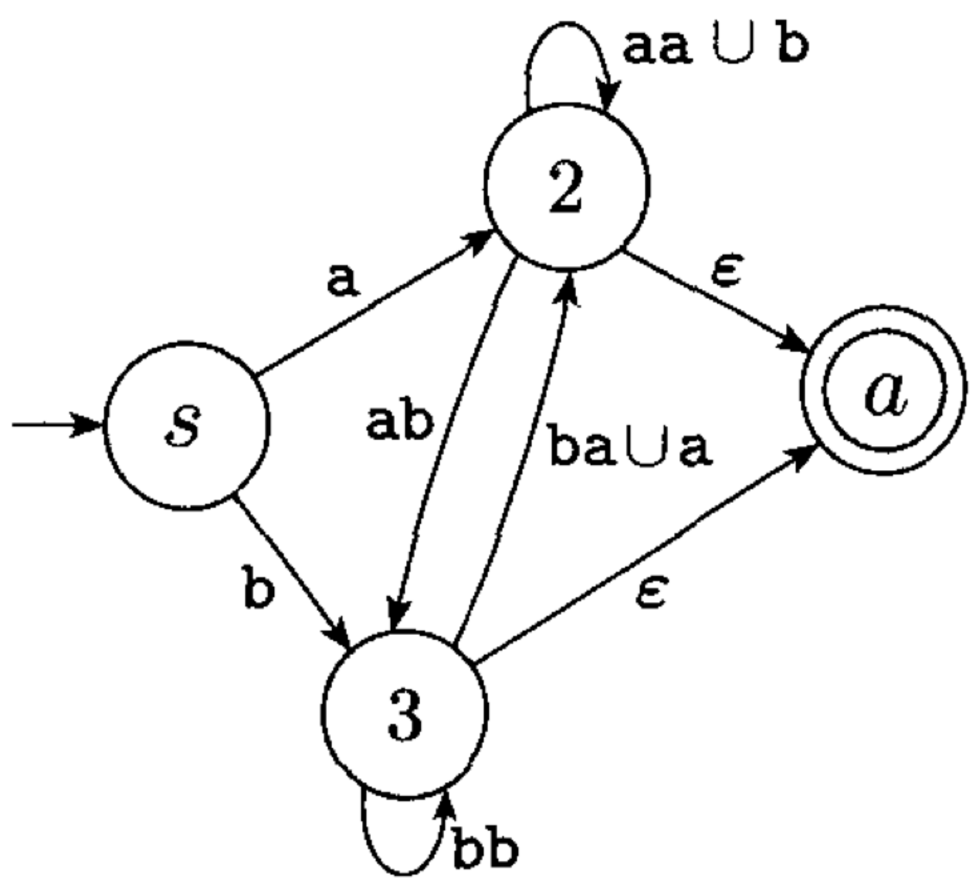

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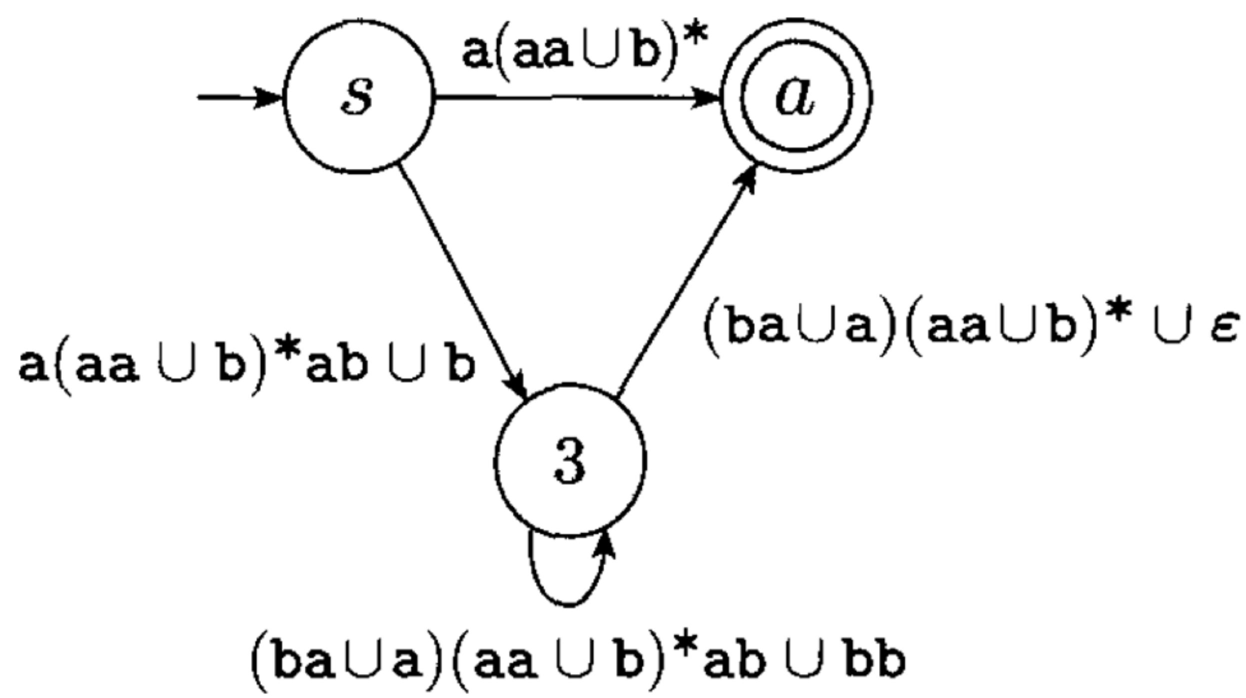

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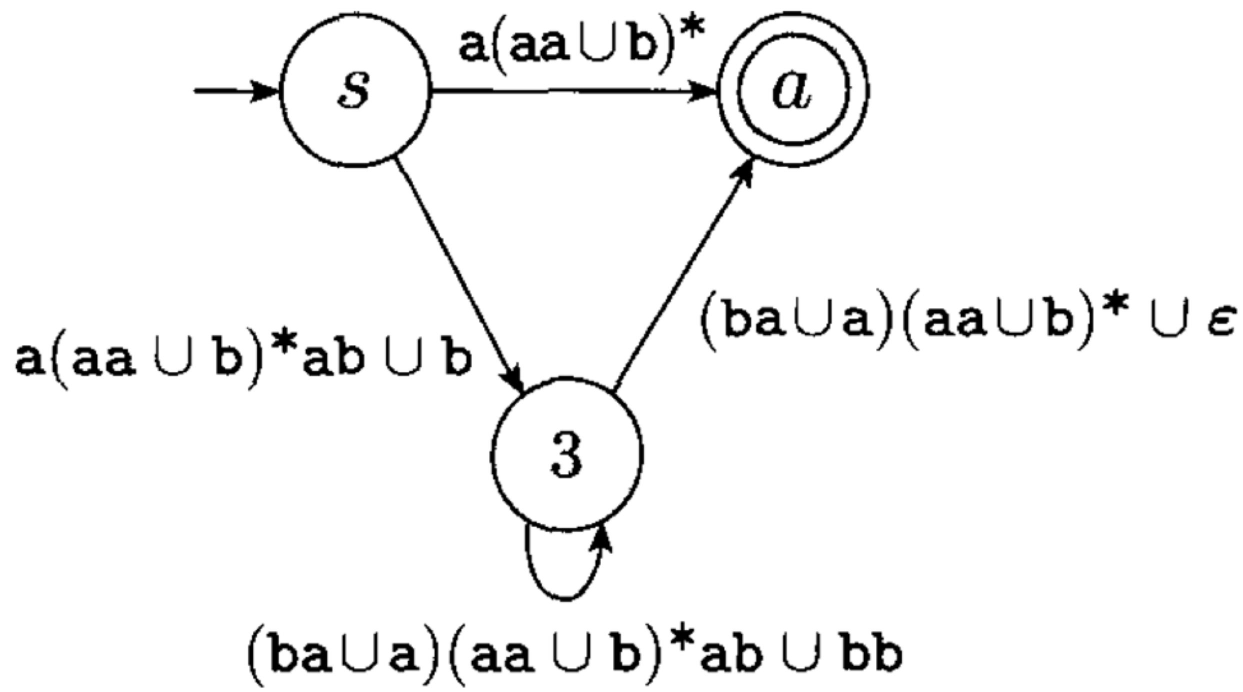

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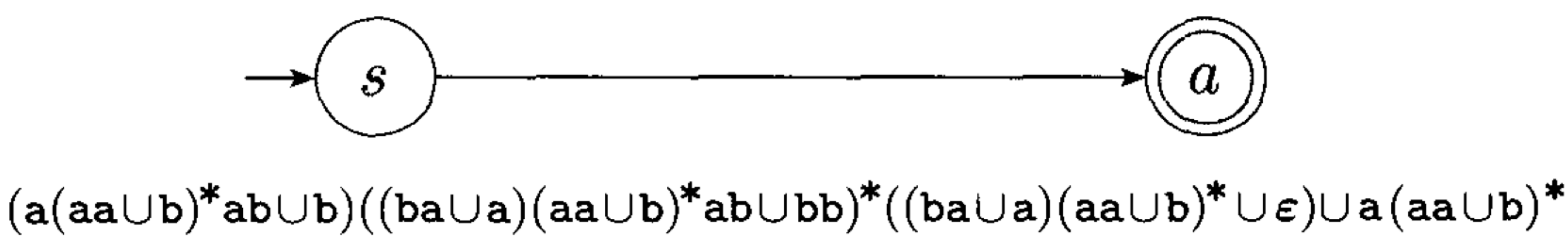
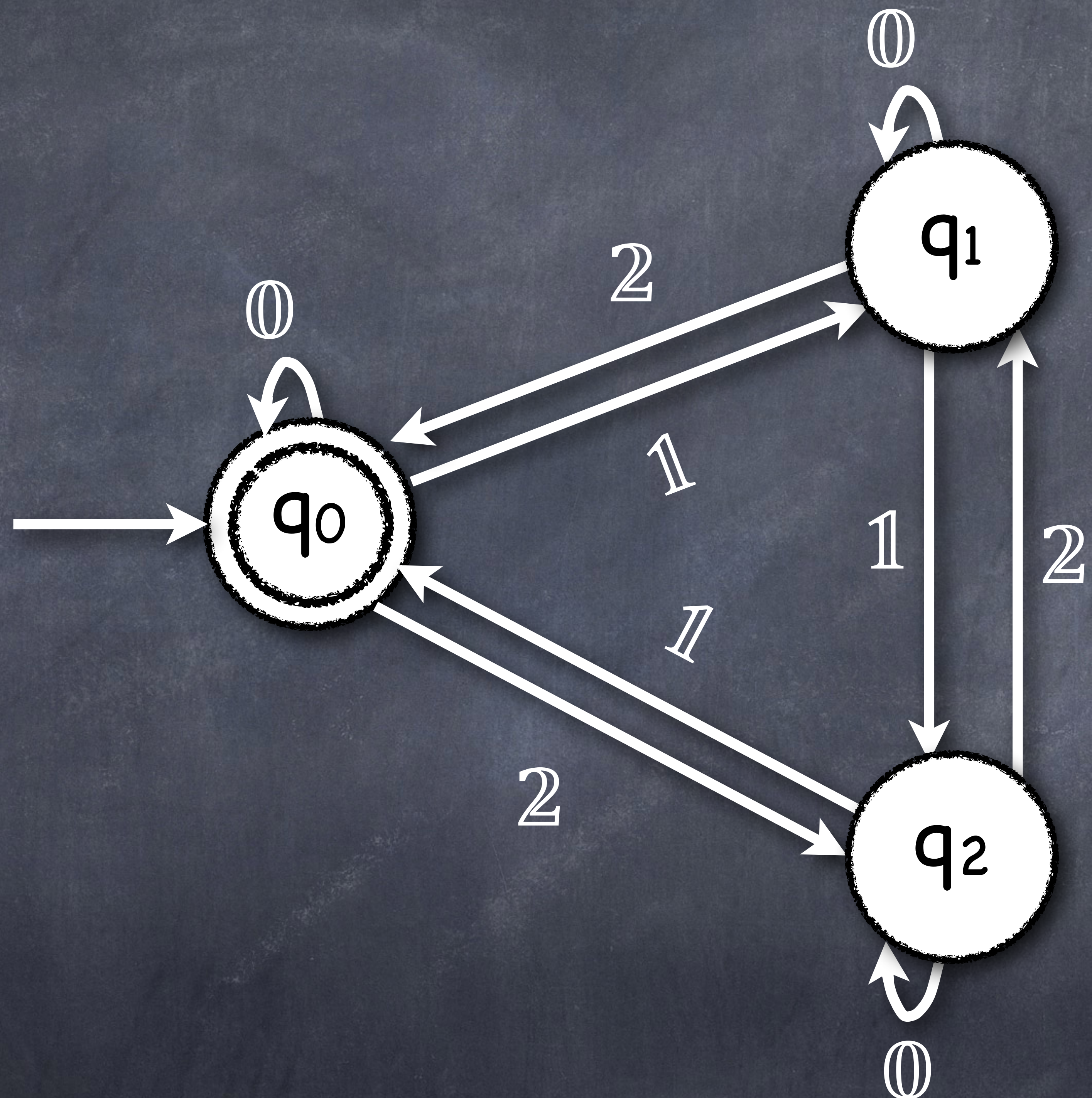


(e)

Multiples of 3 (base 10)



𝟘=0∪3∪6∪9,   𝟙=1∪4∪7,   𝟚=2∪5∪8

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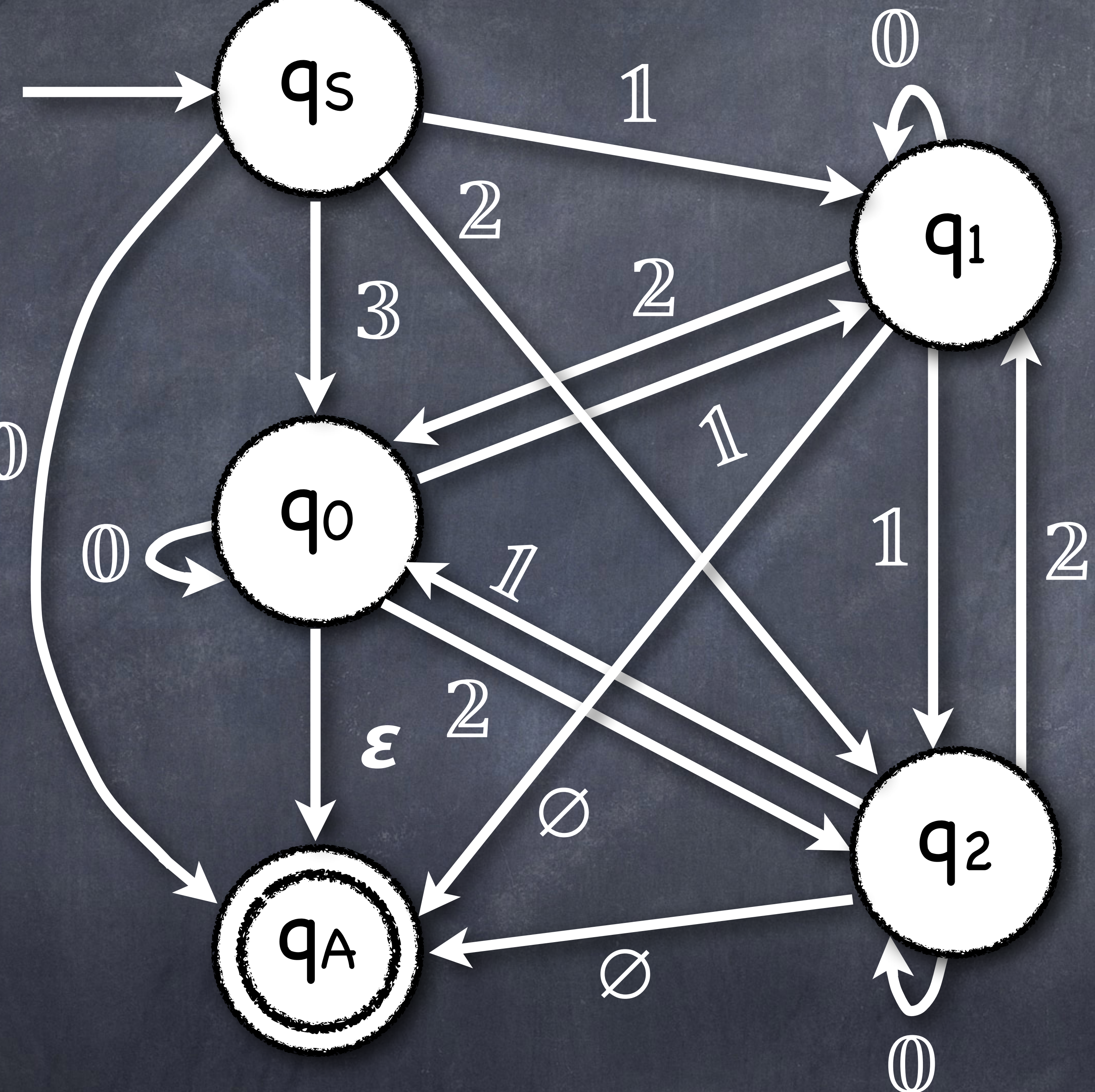

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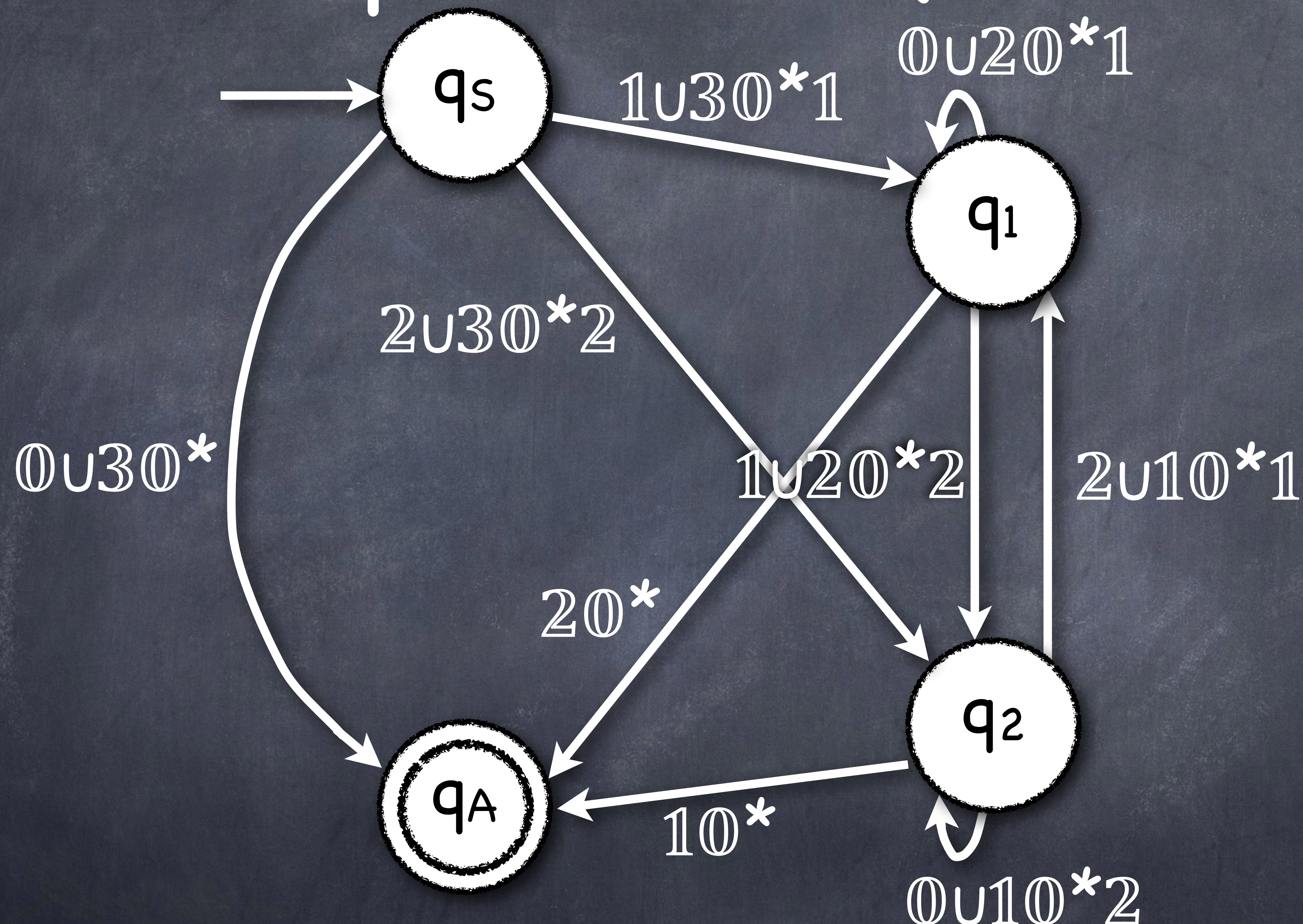

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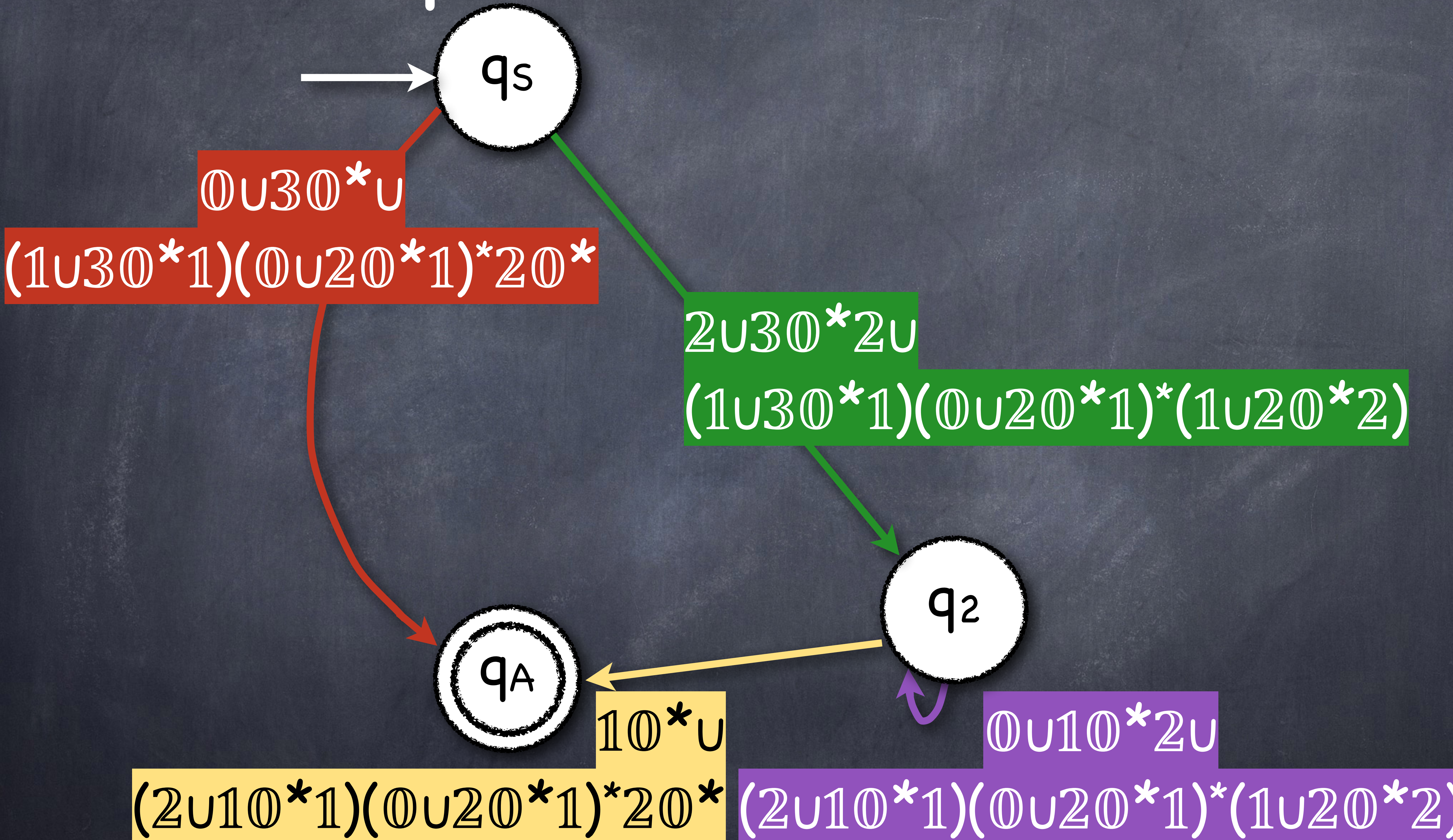

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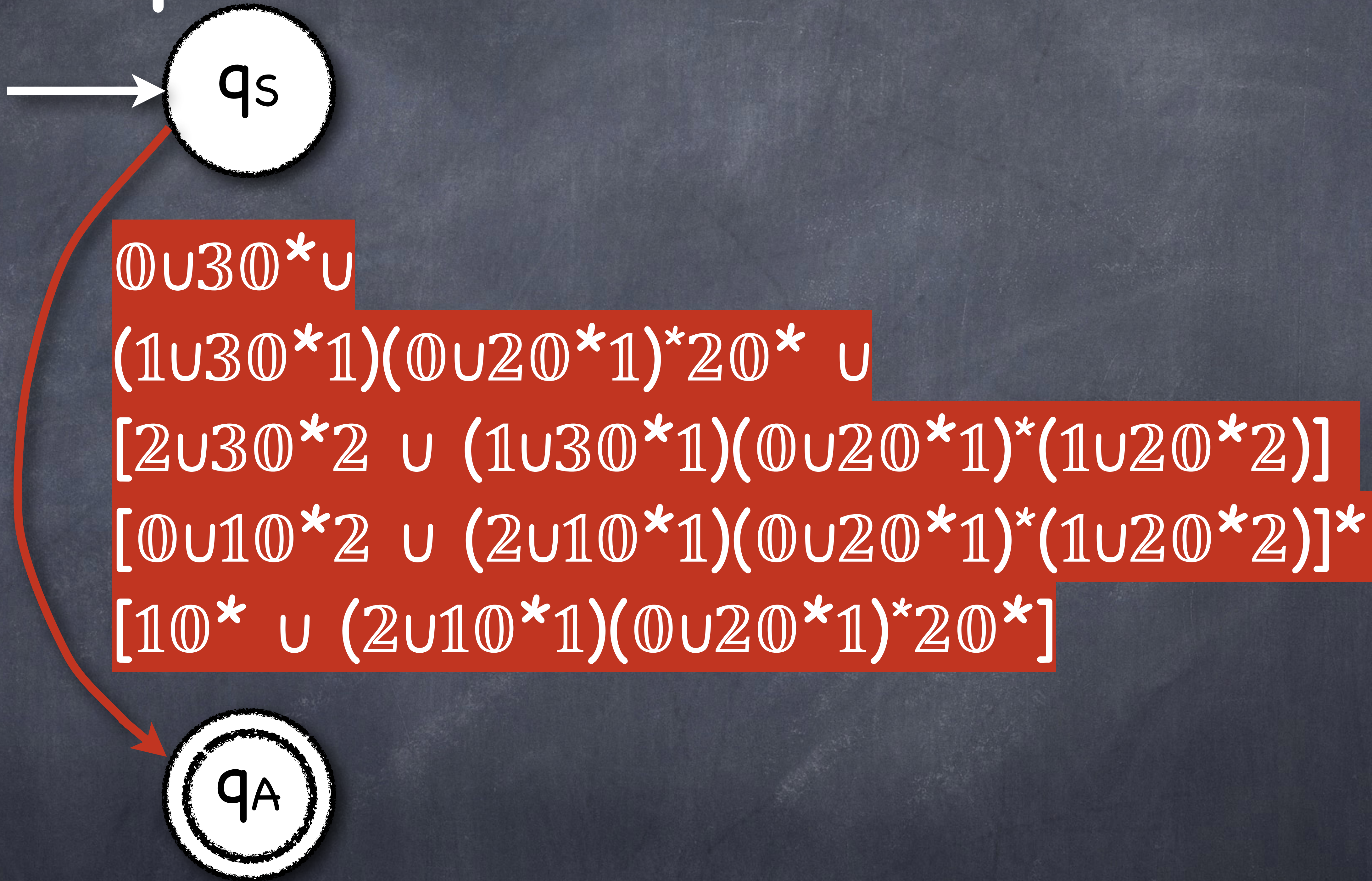

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$$(\mathbb{1} \cup \mathbb{3}\mathbb{0}^*\mathbb{1})(\mathbb{0} \cup \mathbb{2}\mathbb{0}^*\mathbb{1})^*\,\mathbb{2}\mathbb{0}^* \;\cup$$
$$[\mathbb{2} \cup \mathbb{3}\mathbb{0}^*\mathbb{2} \cup (\mathbb{1} \cup \mathbb{3}\mathbb{0}^*\mathbb{1})(\mathbb{0} \cup \mathbb{2}\mathbb{0}^*\mathbb{1})^*(\mathbb{1} \cup \mathbb{2}\mathbb{0}^*\mathbb{2})]$$
$$[\mathbb{0} \cup \mathbb{1}\mathbb{0}^*\mathbb{2} \cup (\mathbb{2} \cup \mathbb{1}\mathbb{0}^*\mathbb{1})(\mathbb{0} \cup \mathbb{2}\mathbb{0}^*\mathbb{1})^*(\mathbb{1} \cup \mathbb{2}\mathbb{0}^*\mathbb{2})]^*$$
$$[\mathbb{1}\mathbb{0}^* \cup (\mathbb{2} \cup \mathbb{1}\mathbb{0}^*\mathbb{1})(\mathbb{0} \cup \mathbb{2}\mathbb{0}^*\mathbb{1})^*\,\mathbb{2}\mathbb{0}^*]$$

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2. Compute equivalent DFAs M and M′ (Thm 1.39).

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3. Using part (b) of Myhill-Nerode we construct minimal DFAs W for M and W′ for M′.

4. L(R)=L(R′) iff W≈W′
   (≈ means "identical up to state renaming").

Regular and non-Regular Languages

footnote 3 page 46:

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- Let $M_A=(Q_A,\Sigma,\delta_A,q_{0A},F_A)$ be a DFA accepting L_A and $M_B=(Q_B,\Sigma,\delta_B,q_{0B},F_B)$ be a DFA accepting L_B.

footnote 3 page 46:

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- Consider $M_U=(Q_A \times Q_B,\Sigma,\delta_U,(q_{0A},q_{0B}),F_U)$ where
  $\delta_U((q,q'),s) = (\delta_A(q,s), \delta_B(q',s))$ for all q,q',s
  and
  $F_U = (FA \times QB) \cup (QA \times FB)$.

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  $F_U = (FA \times QB) \cup (QA \times FB)$.
- $L_U = L_A \cup L_B$.

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  and
  $$F_U = (FA \times QB) \cup (QA \times FB).$$
- $L_U = L_A \cup L_B$.
- $F_U = F_A \times F_B$ would yield the **intersection** (and not the **union**) of L_A and L_B.
  This proves that the class of regular languages is also closed under intersection.

NON-Regular Languages

- $B = \{ 0^n1^n \mid n \geq 0 \}$
- $C = \{ w \mid w$ contains an equal number of 0's and 1's $\}$
- $D = \{ w \mid w$ contains an equal number of occurrences of 01 and 10 as sub-strings $\}$

NON-Regular Languages

NON-Regular

- B … 0 }
- C = { w | w contains an equal number of 0's and 1's }
- D = { w | w contains an equal number of occurrences of 01 and 10 as sub-strings }

NON-Regular Languages

- B [illegible] 0 }

NON-Regular

- C = { w | w c[illegible] equal number of 0's and 1's }

NON-Regular

- D = { w | w contains an equal number of occurrences of 01 and 10 as sub-strings }

NON-Regular Languages

- B … }

NON-Regular

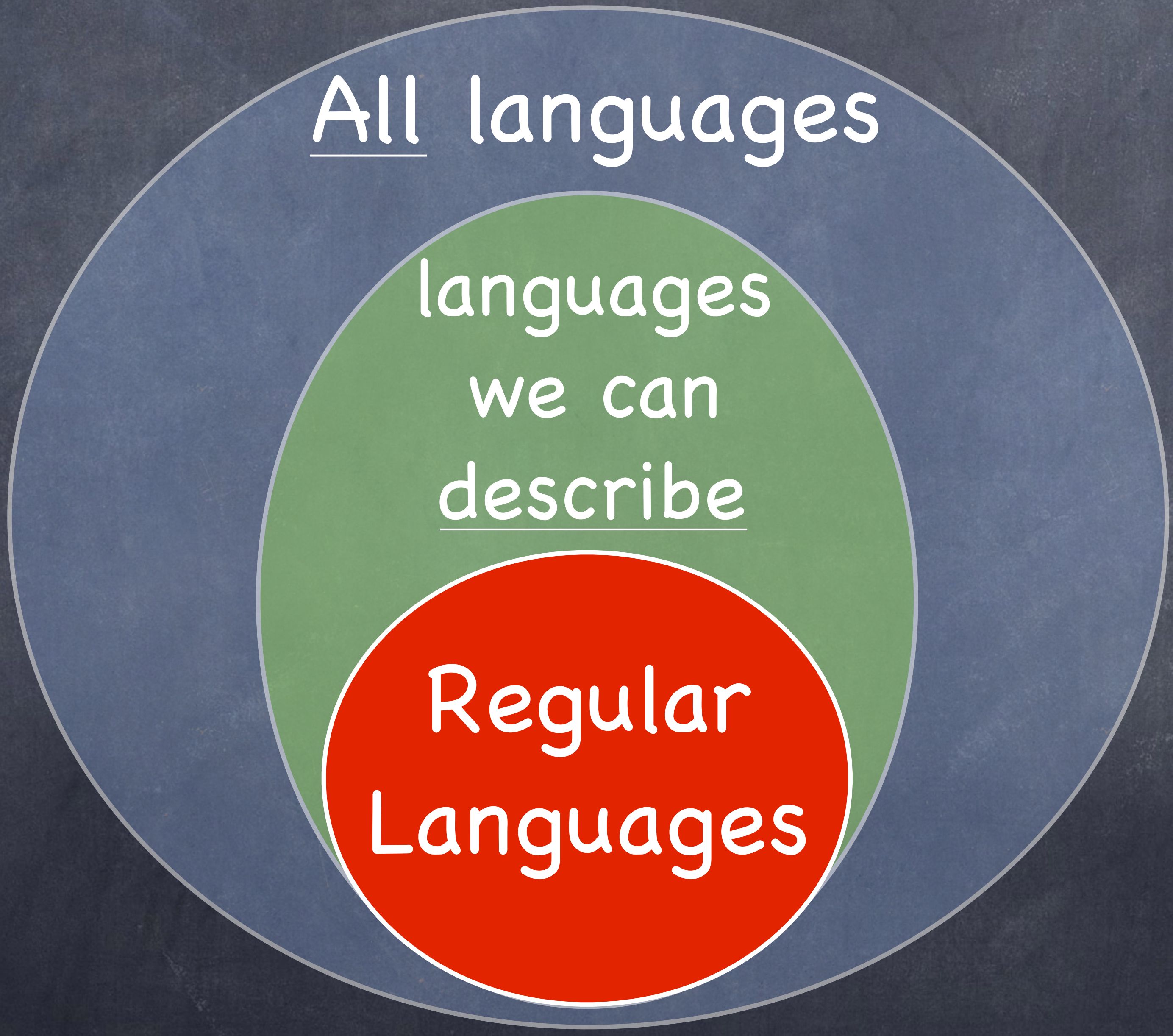
- C = { w | w c… equal number of 0's and 1's }

NON-Regular

- D = { w | w con… equal number of occurrences o… and 10 as sub-strings }

Regular

Computability Theory

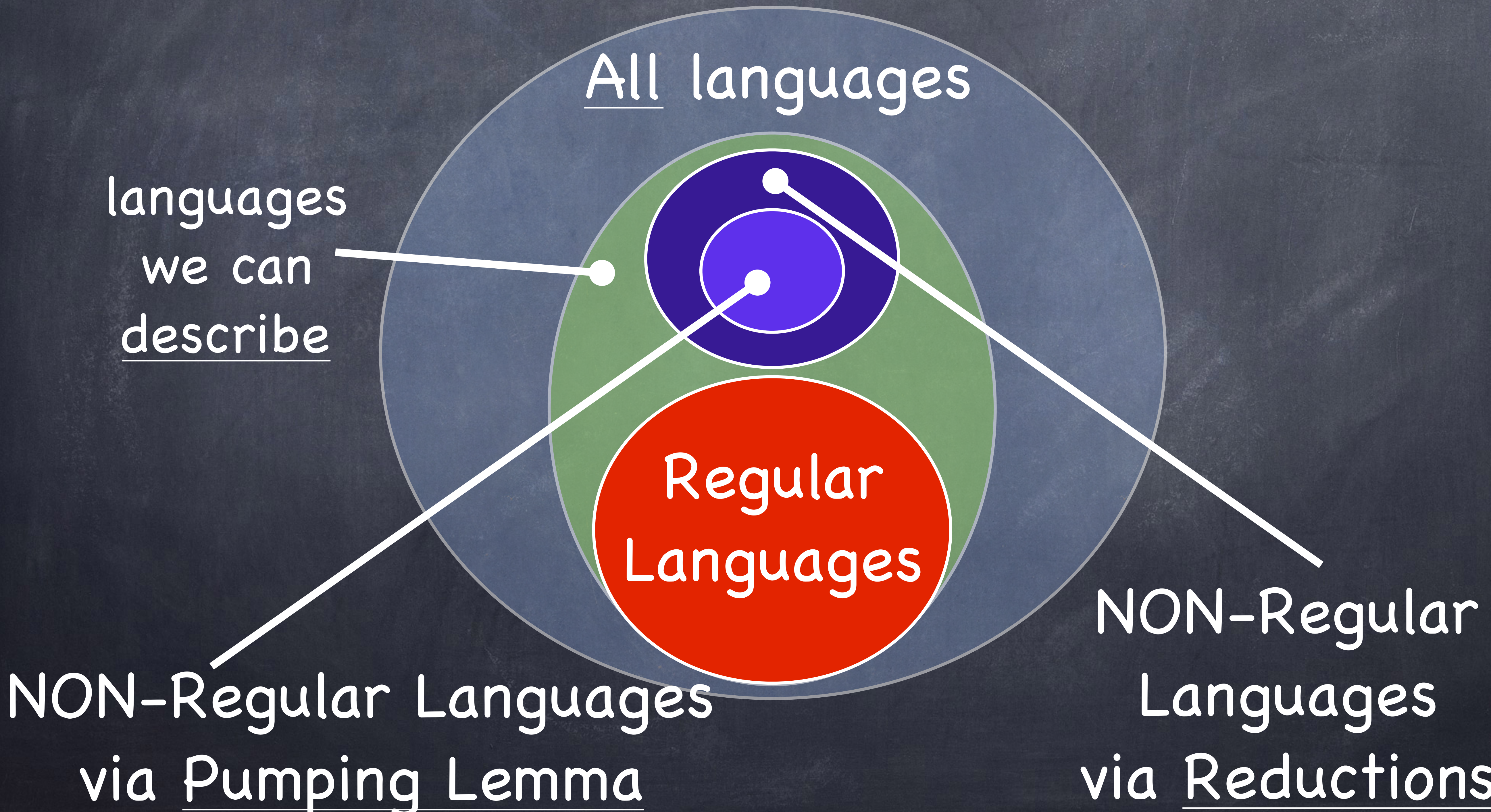


NON-Regular Languages

- Theorem: Some languages are not regular.

  Proof idea: all regular languages have certain properties. Some languages provably do not have one of these properties.

Computability Theory



Reductions

- If C is regular then so is B.
- Proof: Regular languages are closed under intersection (see footnote 3 page 46). Define $A = L(0^*1^*)$. Obviously A is regular. If C was regular then so would $C \cap A = B$.

  QED

- If B is NON-regular then so is C.

$B = \{ 0^n1^n \mid n \geq 0 \}$

$C = \{ w \mid w \text{ contains an equal number of 0's and 1's} \}$

Reductions

- If A is regular then so is A′.
- Regular laguages are closed under complement (see ex. 1.14), intersection, union, concatenation and star. If there exists R, a regular language, such that either $A^c=A'$, $A^*=A'$, $A\cap R=A'$, $A\cup R=A'$, $A\circ R=A'$ or any combinations of these operations then A′ is regular as long as A is.
- If A′ is NON-regular then so is A.

Simple Reductions

- If A^* is NON-regular then so is A.
- If A is NON-regular then so is A^c.
- If A is NON-regular then so is A^R.

Complex Reductions

- Let $A' = (A \cup R) \cap (A^c \cup R')$ (R,R′ regular)
- Let $A' = ((A^c \cap R) \cup (A^* \cap R')) \circ R''$ (R,R′,R″ regular)
- Let $A' = (A \circ R) \cap (A^c \circ R')$ (R,R′ regular)
- If A′ is NON-regular then so is A.

NON-Regular Languages

- Theorem: Some languages are not regular.

  Proof idea: all regular languages have certain properties. Some languages provably do not have one of these properties.

- Example: A property of all regular languages = the Pumping Lemma.

COMP-330
Theory of Computation

Fall 2019 -- Prof. Claude Crépeau

Lec. 8 : Regular and NON-Reg. Languages