# COMP-330
# Theory of Computation

Fall 2019 -- Prof. Claude Crépeau

# Lec. 13 :
# Pumping Lemma for CFLs

Announcements

## EXCEPTIONAL OFFICE HOURS

Posted Oct 14, 2019 11:44 AM

On Wednesday (Oct 16) I will be attending an external meeting during most of the day. My office hours will exceptionally be held from 11:00 to 14:00, same place as usual, McConnell 110N.

Claude

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## Office Hours

Posted Oct 11, 2019 6:06 PM

Hi,

As next Monday is a holiday. I will have the OH for next week on Wednesday between 11.30am - 12.30pm.

Regards,

Anirudha

# PDA to CFG

**PROOF** Say that $P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{accept}}\})$ and construct $G$. The variables of $G$ are $\{A_{pq} | \, p, q \in Q\}$. The start variable is $A_{q_0, q_{\text{accept}}}$. Now we describe $G$'s rules.

- For each $p, q, r, s \in Q$, $t \in \Gamma$, and $a, b \in \Sigma_\varepsilon$, if $\delta(p, a, \varepsilon)$ contains $(r, t)$ and $\delta(s, b, t)$ contains $(q, \varepsilon)$, put the rule $A_{pq} \rightarrow a A_{rs} b$ in $G$.
- For each $p, q, r \in Q$, put the rule $A_{pq} \rightarrow A_{pr} A_{rq}$ in $G$.
- Finally, for each $p \in Q$, put the rule $A_{pp} \rightarrow \varepsilon$ in $G$.

# PDA to CFG

**CLAIM 2.30**

If $A_{pq}$ generates $x$, then $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack).

We prove this claim by induction on the number of steps in the derivation of $x$ from $A_{pq}$.

If $A_{pq}$ generates $x$, then $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack).

***Basis:*** The derivation has 1 step.

A derivation with a single step must use a rule whose right-hand side contains no variables. The only rules in $G$ where no variables occur on the right-hand side are $A_{pp} \to \varepsilon$. Clearly, input $\varepsilon$ takes $P$ from $p$ with empty stack to $p$ with empty stack so the basis is proved.

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***Induction step:*** Assume true for derivations of length at most $k$, where $k \geq 1$, and prove true for derivations of length $k + 1$.

If $A_{pq}$ generates $x$, then $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack).

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***Induction step:*** Assume true for derivations of length at most $k$, where $k \geq 1$, and prove true for derivations of length $k+1$.

Suppose that $A_{pq} \overset{*}{\Rightarrow} x$ with $k+1$ steps. The first step in this derivation is either $A_{pq} \Rightarrow aA_{rs}b$ or $A_{pq} \Rightarrow A_{pr}A_{rq}$. We handle these two cases separately.

If $A_{pq}$ generates $x$, then $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack).

In the first case, consider the portion $y$ of $x$ that $A_{rs}$ generates, so $x = ayb$. Because $A_{rs} \overset{*}{\Rightarrow} y$ with $k$ steps, the induction hypothesis tells us that $P$ can go from $r$ on empty stack to $s$ on empty stack. Because $A_{pq} \rightarrow aA_{rs}b$ is a rule of $G$, $\delta(p, a, \varepsilon)$ contains $(r, t)$ and $\delta(s, b, t)$ contains $(q, \varepsilon)$, for some stack symbol $t$. Hence, if $P$ starts at $p$ with an empty stack, after reading $a$ it can go to state $r$ and push $t$ onto the stack. Then reading string $y$ can bring it to $s$ and leave $t$ on the stack. Then after reading $b$ it can go to state $q$ and pop $t$ off the stack. Therefore $x$ can bring it from $p$ with empty stack to $q$ with empty stack.

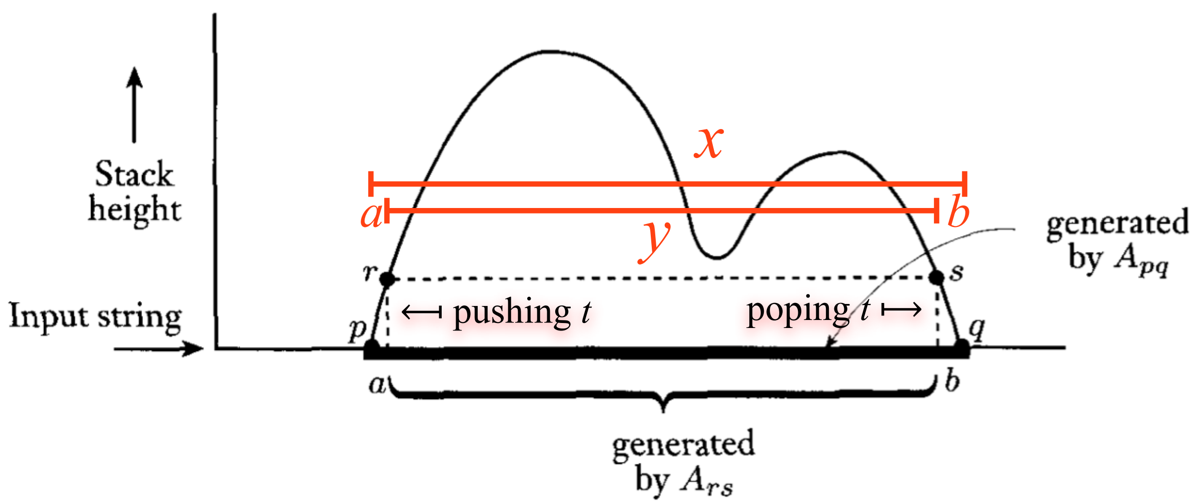


**FIGURE 2.29**
PDA computation corresponding to the rule $A_{pq} \rightarrow aA_{rs}b$

If $A_{pq}$ generates $x$, then $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack).

$A_{pq} \Rightarrow aA_{rs}b$

In the first case, consider the portion $y$ of $x$ that $A_{rs}$ generates, so $x = ayb$. Because $A_{rs} \overset{*}{\Rightarrow} y$ with $k$ steps, the induction hypothesis tells us that $P$ can go from $r$ on empty stack to $s$ on empty stack. Because $A_{pq} \to aA_{rs}b$ is a rule of $G$, $\delta(p, a, \varepsilon)$ contains $(r, t)$ and $\delta(s, b, t)$ contains $(q, \varepsilon)$, for some stack symbol $t$. Hence, if $P$ starts at $p$ with an empty stack, after reading $a$ it can go to state $r$ and push $t$ onto the stack. Then reading string $y$ can bring it to $s$ and leave $t$ on the stack. Then after reading $b$ it can go to state $q$ and pop $t$ off the stack. Therefore $x$ can bring it from $p$ with empty stack to $q$ with empty stack.

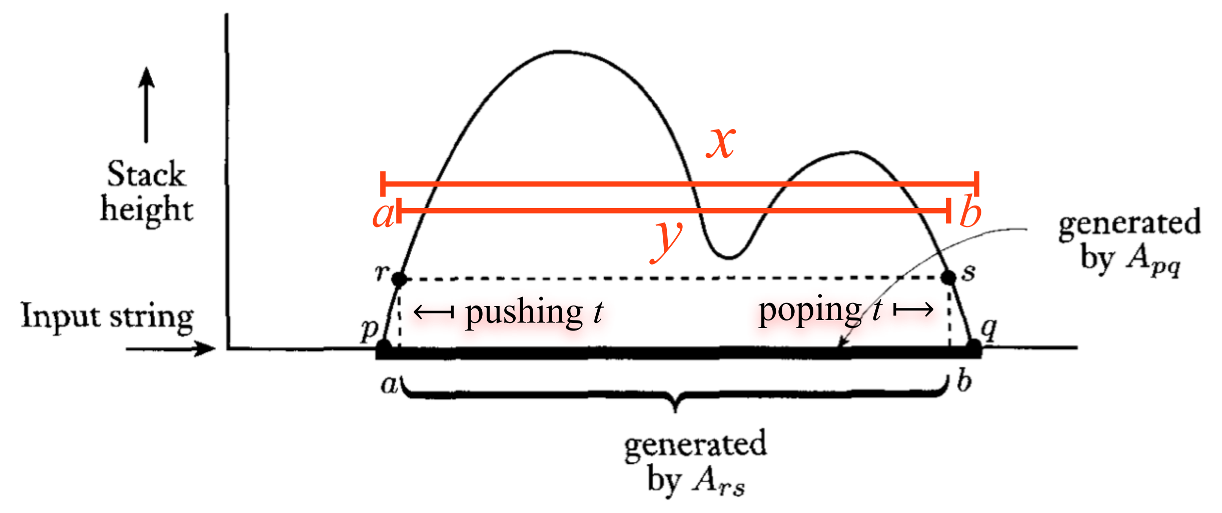


**FIGURE 2.29**
PDA computation corresponding to the rule $A_{pq} \to aA_{rs}b$

If $A_{pq}$ generates $x$, then $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack).

In the second case, consider the portions $y$ and $z$ of $x$ that $A_{pr}$ and $A_{rq}$ respectively generate, so $x = yz$. Because $A_{pr} \overset{*}{\Rightarrow} y$ in at most $k$ steps and $A_{rq} \overset{*}{\Rightarrow} z$ in at most $k$ steps, the induction hypothesis tells us that $y$ can bring $P$ from $p$ to $r$, and $z$ can bring $P$ from $r$ to $q$, with empty stacks at the beginning and end. Hence $x$ can bring it from $p$ with empty stack to $q$ with empty stack. This completes the induction step.

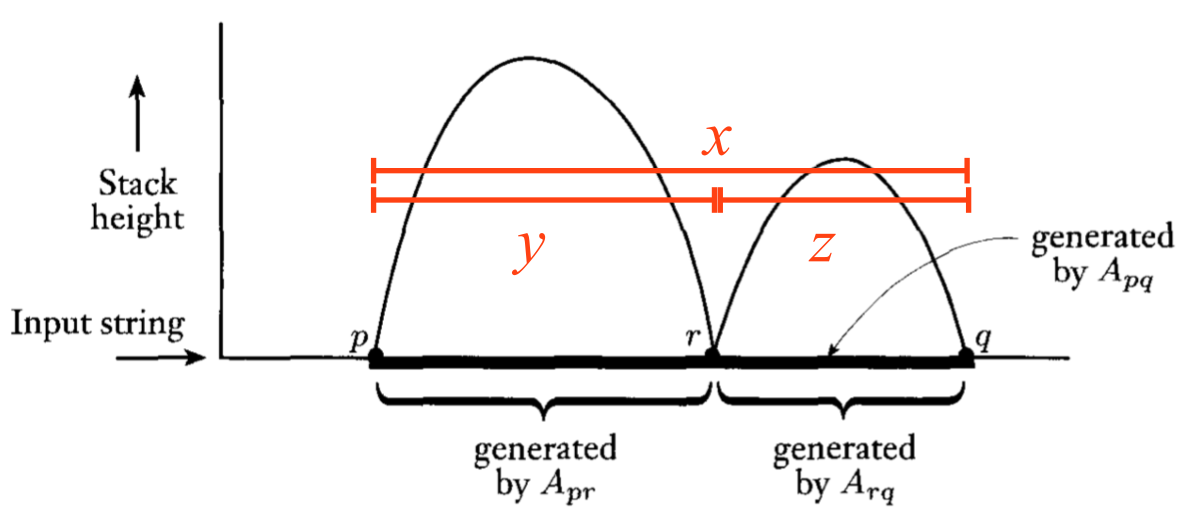


**FIGURE 2.28**
PDA computation corresponding to the rule $A_{pq} \rightarrow A_{pr}A_{rq}$

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$A_{pq} \Rightarrow A_{pr} A_{rq}$

In the second case, consider the portions $y$ and $z$ of $x$ that $A_{pr}$ and $A_{rq}$ respectively generate, so $x = yz$. Because $A_{pr} \overset{*}{\Rightarrow} y$ in at most $k$ steps and $A_{rq} \overset{*}{\Rightarrow} z$ in at most $k$ steps, the induction hypothesis tells us that $y$ can bring $P$ from $p$ to $r$, and $z$ can bring $P$ from $r$ to $q$, with empty stacks at the beginning and end. Hence $x$ can bring it from $p$ with empty stack to $q$ with empty stack. This completes the induction step.

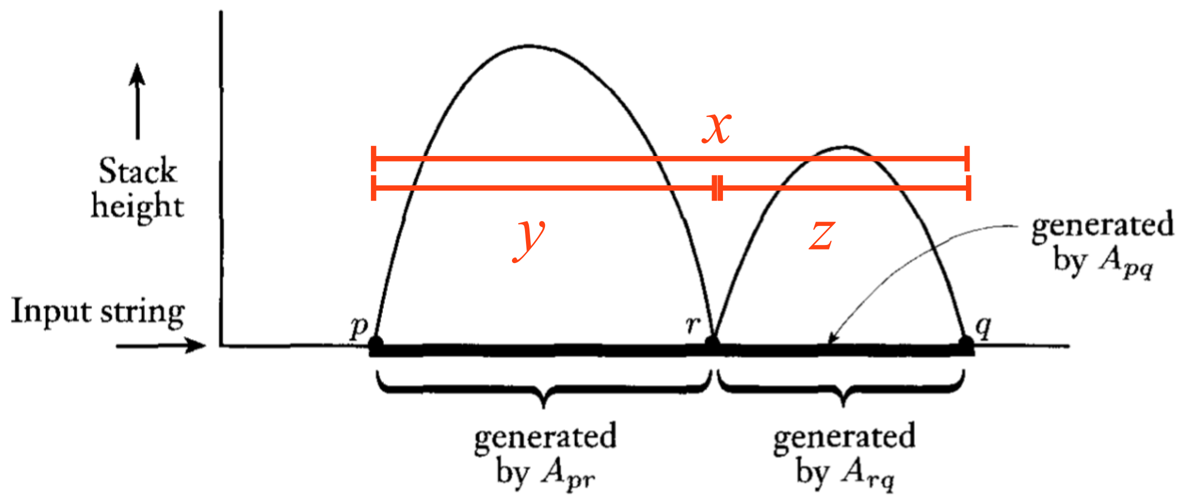


**FIGURE 2.28**
PDA computation corresponding to the rule $A_{pq} \rightarrow A_{pr} A_{rq}$

# PDA to CFG

**CLAIM 2.31**

If $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack), then $A_{pq}$ generates $x$.

We prove this claim by induction on the number of steps in the computation of $P$ that goes from $p$ to $q$ with empty stacks on input $x$.

If $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack), then $A_{pq}$ generates $x$.

***Basis:*** The computation has 0 steps.
If a computation has 0 steps, it starts and ends at the same state—say, $p$. So we must show that $A_{pp} \overset{*}{\Rightarrow} x$. In 0 steps, $P$ only has time to read the empty string, so $x = \varepsilon$. By construction, $G$ has the rule $A_{pp} \rightarrow \varepsilon$, so the basis is proved.

If $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack), then $A_{pq}$ generates $x$.

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***Induction step:*** Assume true for computations of length at most $k$, where $k \geq 0$, and prove true for computations of length $k + 1$.

If $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack), then $A_{pq}$ generates $x$.

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***Induction step:*** Assume true for computations of length at most $k$, where $k \geq 0$, and prove true for computations of length $k + 1$.

Suppose that $P$ has a computation wherein $x$ brings $p$ to $q$ with empty stacks in $k + 1$ steps. Either the stack is empty only at the beginning and end of this computation, or it becomes empty elsewhere, too.

If $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack), then $A_{pq}$ generates $x$.

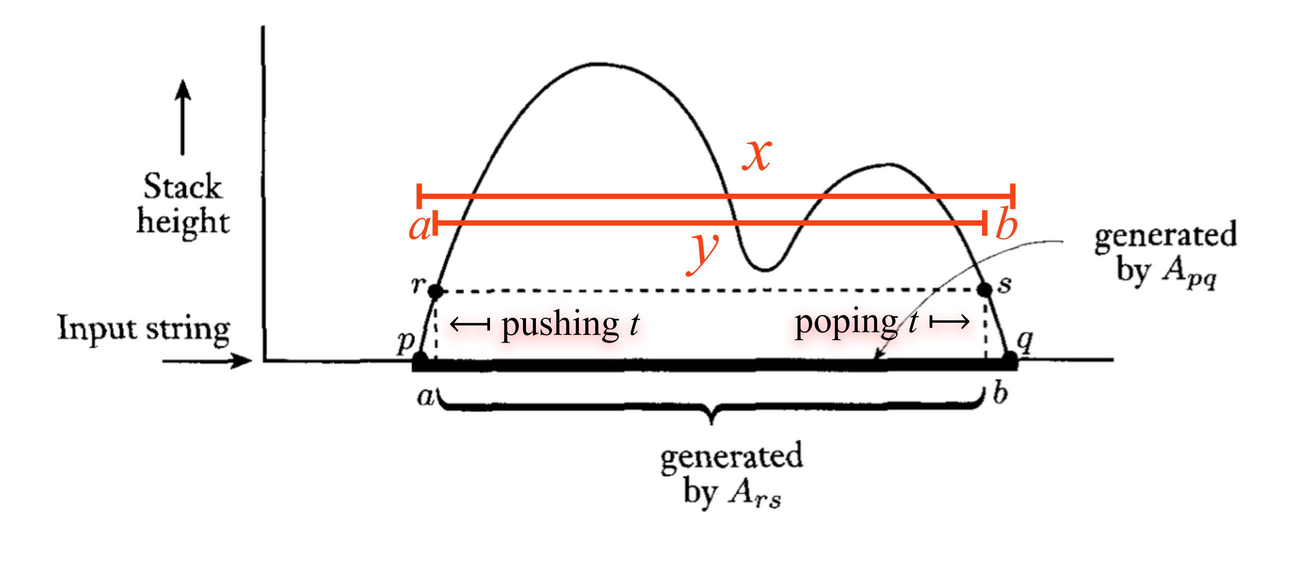
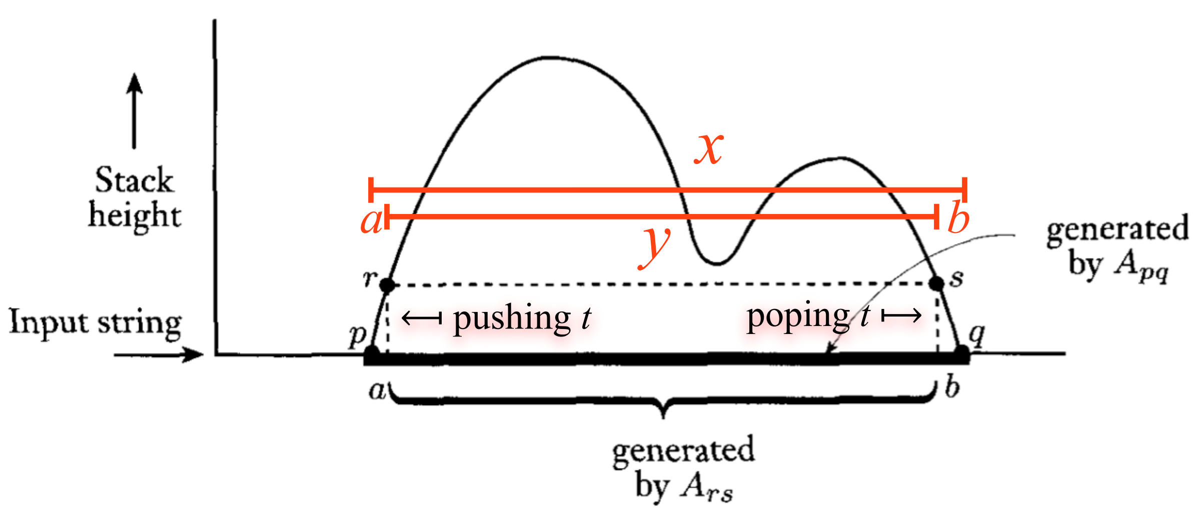



**FIGURE 2.29**
PDA computation corresponding to the rule $A_{pq} \to aA_{rs}b$

If $x$ can bring $P$ from a state $p$ (with an empty stack) to a state $q$ (with an empty stack), then $A_{pq}$ generates $x$.

In the first case, the symbol that is pushed at the first move must be the same as the symbol that is popped at the last move. Call this symbol $t$. Let $a$ be the input read in the first move, $b$ be the input read in the last move, $r$ be the state after the first move, and $s$ be the state before the last move. Then $\delta(p, a, \varepsilon)$ contains $(r, t)$ and $\delta(s, b, t)$ contains $(q, \varepsilon)$, and so rule $A_{pq} \to aA_{rs}b$ is in $G$.

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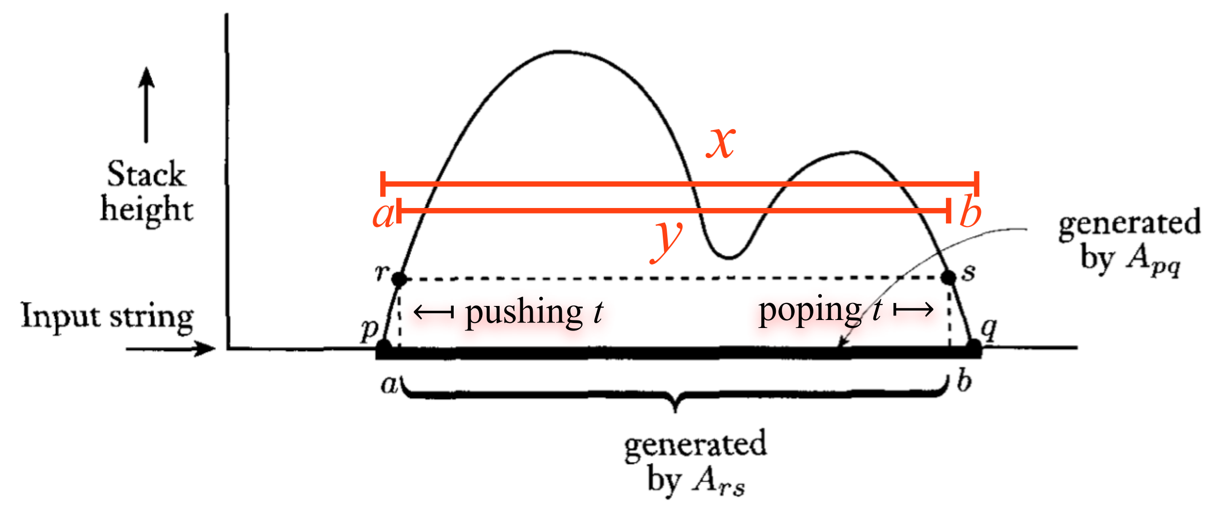


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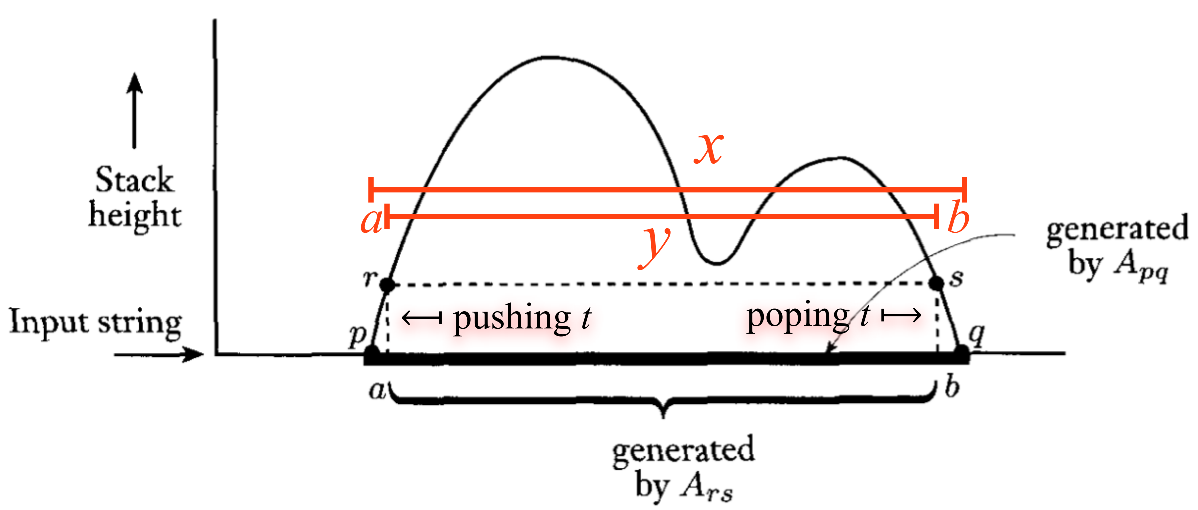
- For each $p, q, r, s \in Q$, $t \in \Gamma$, and $a, b \in \Sigma_\varepsilon$, if $\delta(p, a, \varepsilon)$ contains $(r, t)$ and $\delta(s, b, t)$ contains $(q, \varepsilon)$, put the rule $A_{pq} \to aA_{rs}b$ in $G$.

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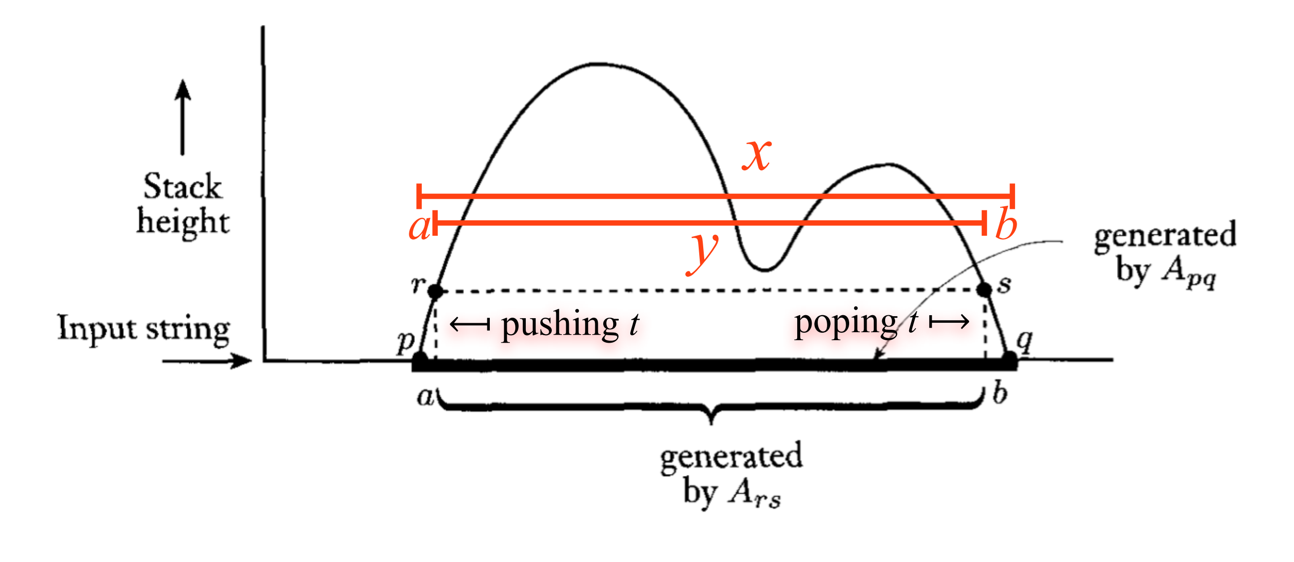


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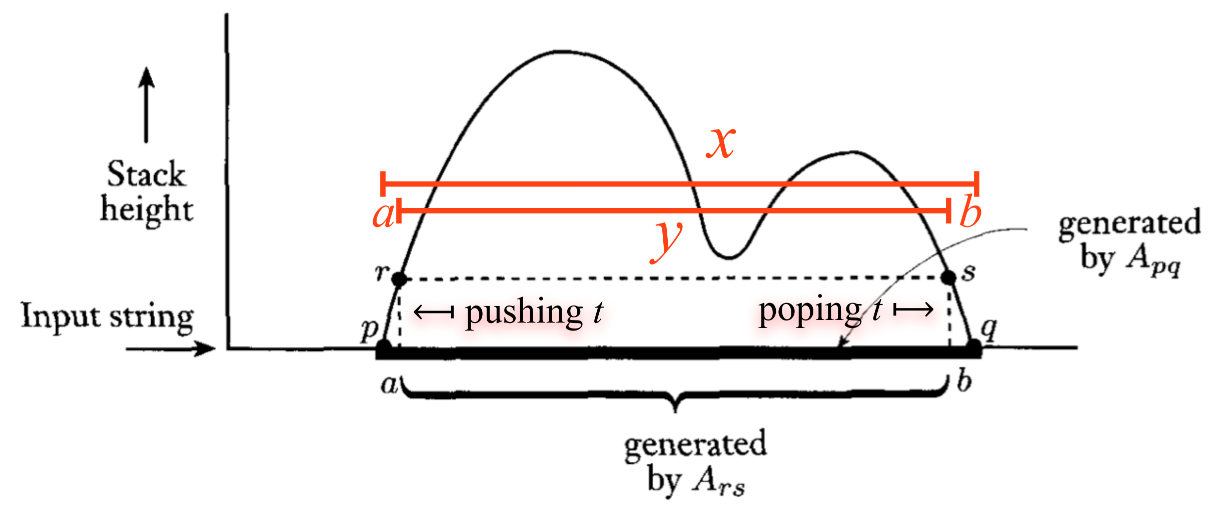
Let $y$ be the portion of $x$ without $a$ and $b$, so $x = ayb$. Input $y$ can bring $P$ from $r$ to $s$ without touching the symbol $t$ that is on the stack and so $P$ can go from $r$ with an empty stack to $s$ with an empty stack on input $y$. We have removed the first and last steps of the $k+1$ steps in the original computation on $x$ so the computation on $y$ has $(k+1) - 2 = k - 1$ steps. Thus the induction hypothesis tells us that $A_{rs} \overset{*}{\Rightarrow} y$. Hence $A_{pq} \overset{*}{\Rightarrow} x$.

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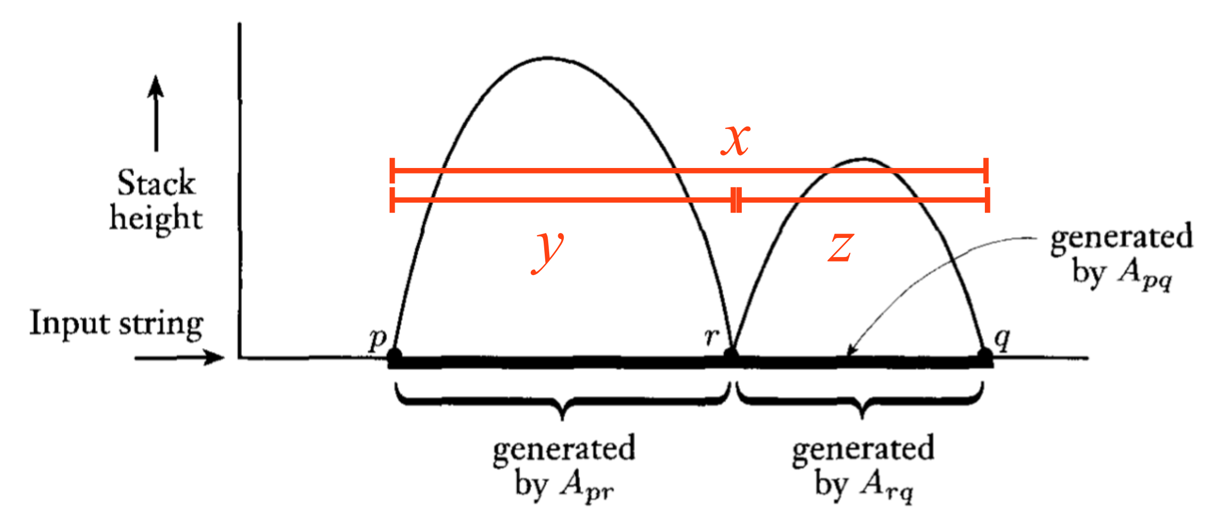
In the second case, let $r$ be a state where the stack becomes empty other than at the beginning or end of the computation on $x$. Then the portions of the computation from $p$ to $r$ and from $r$ to $q$ each contain at most $k$ steps. Say that $y$ is the input read during the first portion and $z$ is the input read during the second portion. The induction hypothesis tells us that $A_{pr} \overset{*}{\Rightarrow} y$ and $A_{rq} \overset{*}{\Rightarrow} z$. Because rule $A_{pq} \rightarrow A_{pr}A_{rq}$ is in $G$, $A_{pq} \overset{*}{\Rightarrow} x$, and the proof is complete.

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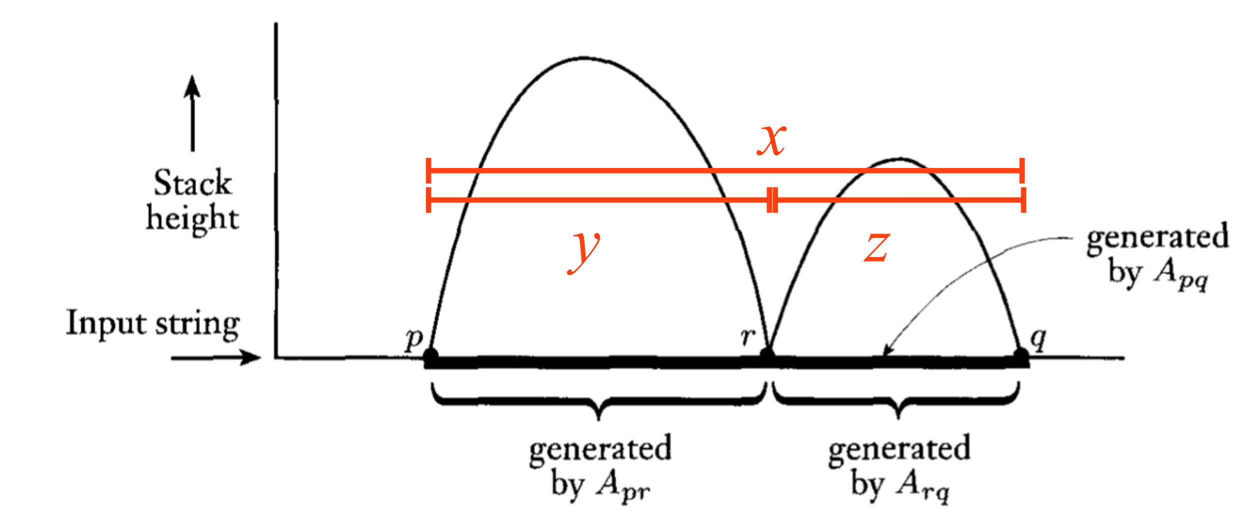


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# PDA vs CFG

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**LEMMA 2.21**

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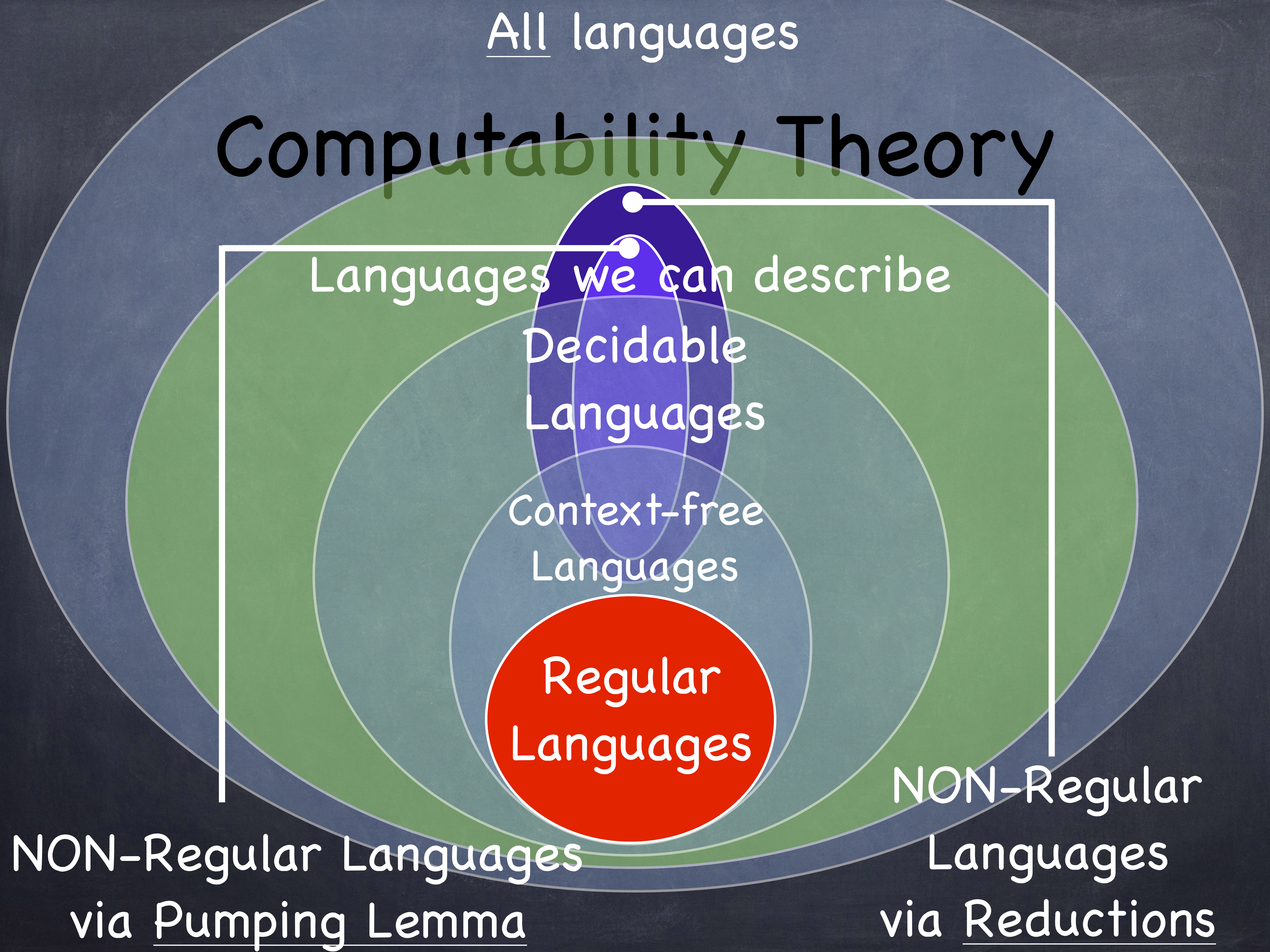
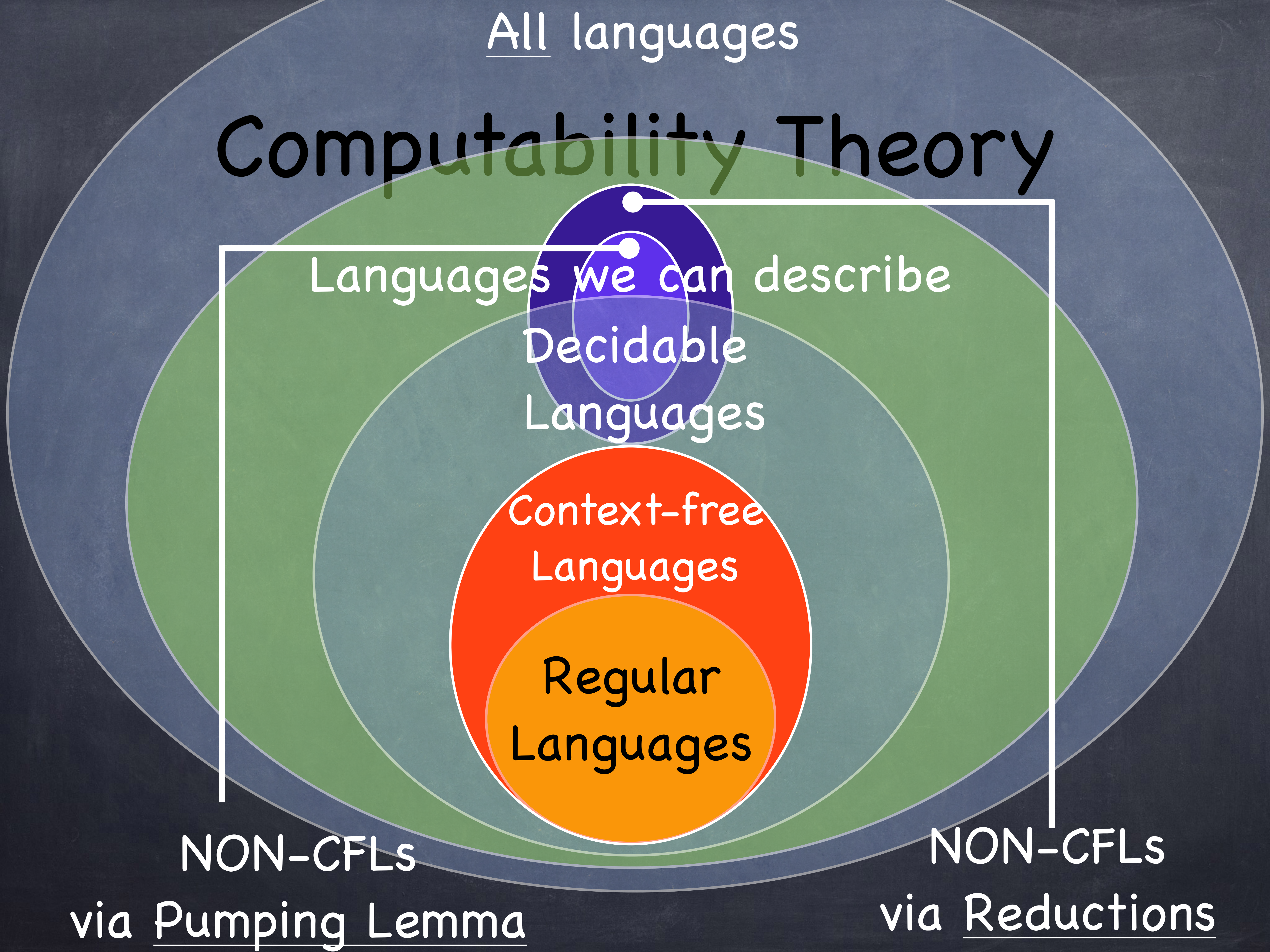
**THEOREM 2.20**

A language is context free if and only if some pushdown automaton recognizes it.

# Computability Theory




All languages
Computability Theory
Languages we can describe
Decidable Languages
Context-free Languages
Regular Languages
NON-CFLs via Pumping Lemma
NON-CFLs via Reductions

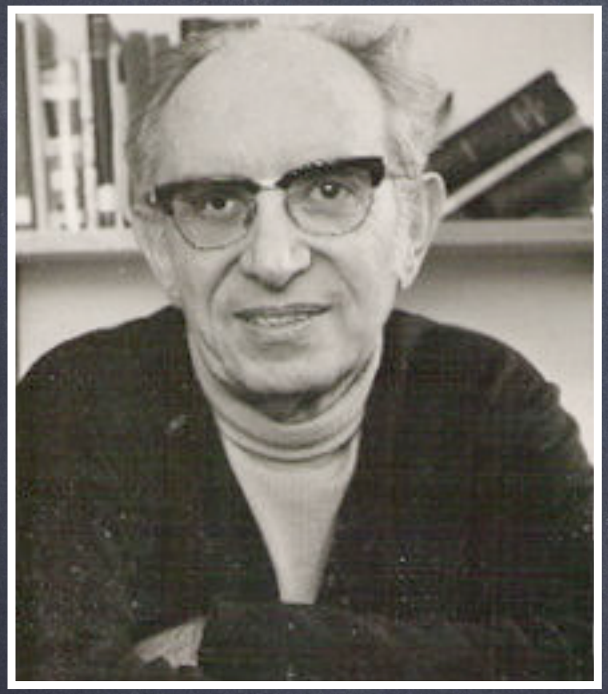
# Pumping Lemma for CFLs

**THEOREM 2.34**

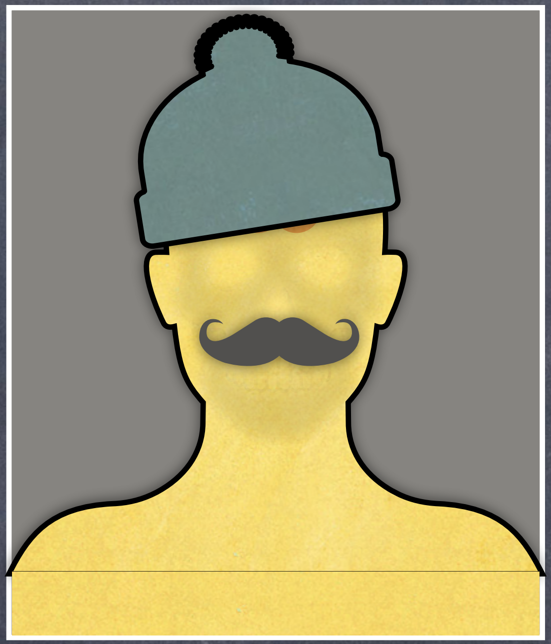
**Pumping lemma for context-free languages** If $A$ is a context-free language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

# Pumping Lemma for CFLs



Yehoshua Bar-Hillel



Micha A. Perles

Eli Shamir

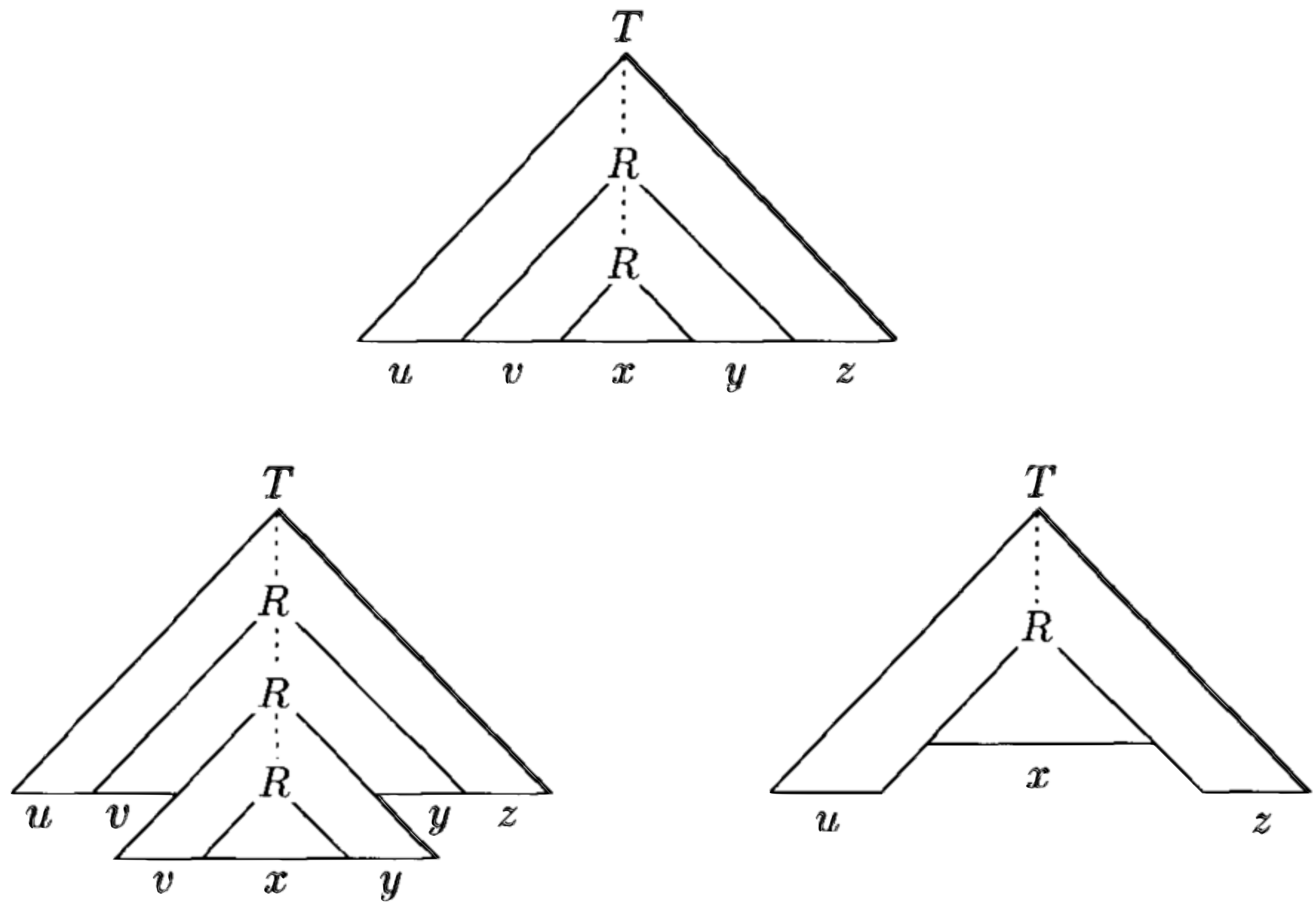



**FIGURE 2.35**
Surgery on parse trees

# Pumping Lemma

**PROOF** Let $G$ be a CFG for CFL $A$. Let $b$ be the maximum number of symbols in the right-hand side of a rule. In any parse tree using this grammar we know that a node can have no more than $b$ children. In other words, at most $b$ leaves are 1 step from the start variable; at most $b^2$ leaves are within 2 steps of the start variable; and at most $b^h$ leaves are within $h$ steps of the start variable. So, if the height of the parse tree is at most $h$, the length of the string generated is at most $b^h$. **Reciprocally,** if a generated string is at least $b^h + 1$ long, each of its parse trees must be at least $h + 1$ high.

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Say $|V|$ is the number of variables in $G$. We set $p$, the pumping length, to be $b^{|V|+1}$. Now if $s$ is a string in $A$ and its length is $p$ or more, its parse tree must be at least $|V| + 1$ high, **because $b^{|V|+1} \geq b^{|V|}+1$.**

# Pumping Lemma

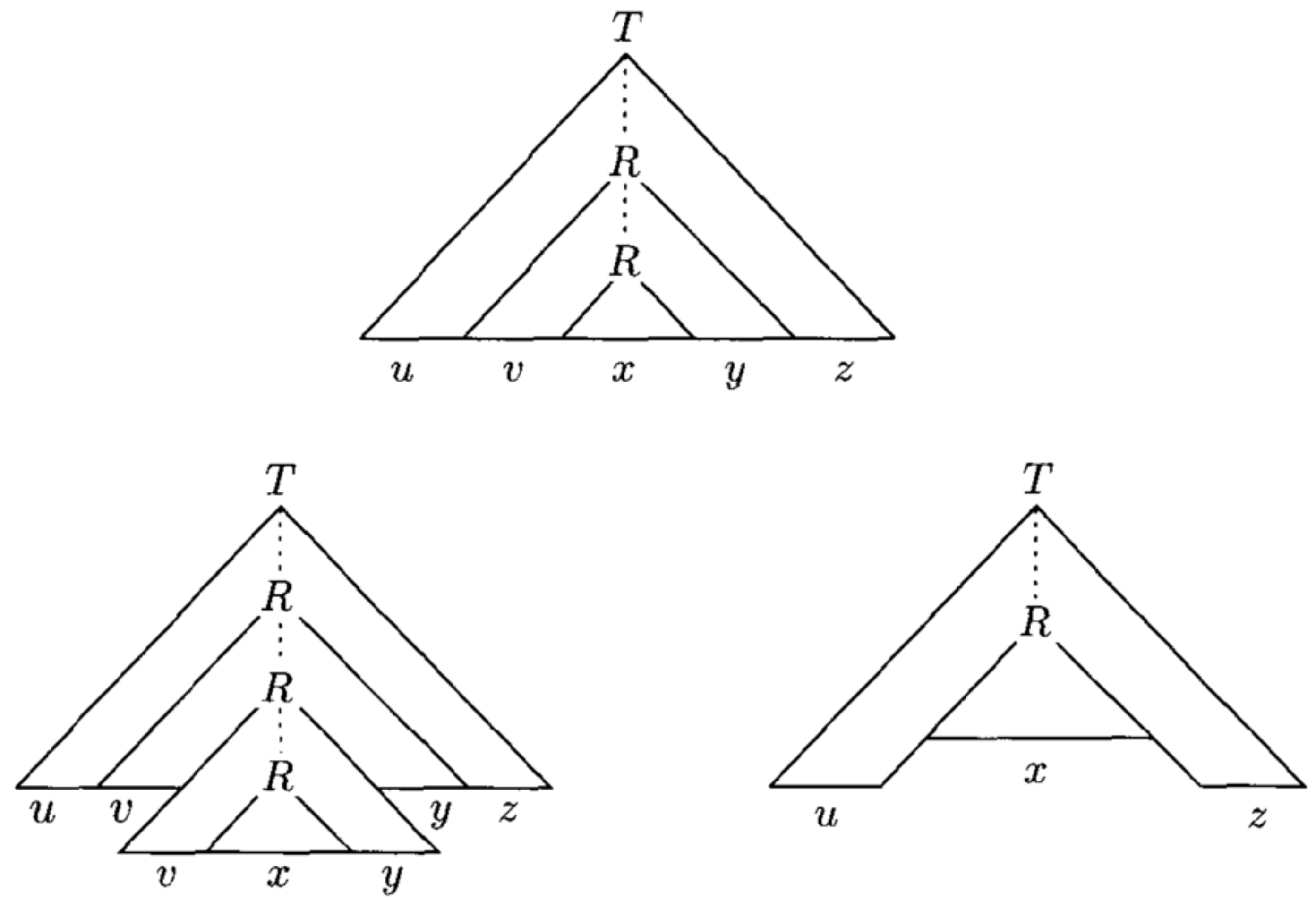


**FIGURE 2.35**
Surgery on parse trees

To see how to pump any such string $s$, let $\tau$ be one of its parse trees. If $s$ has several parse trees, choose $\tau$ to be a parse tree that has the smallest number of nodes. We know that $\tau$ must be at least $|V| + 1$ high, so it must contain a path from the root to a leaf of length at least $|V| + 1$. That path has at least $|V| + 2$ nodes; one at a terminal, the others at variables. Hence that path has at least $|V| + 1$ variables. With $G$ having only $|V|$ variables, some variable $R$ appears more than once on that path. For convenience later, we select $R$ to be a variable that repeats among the lowest $|V| + 1$ variables on this path.

# Pumping Lemma

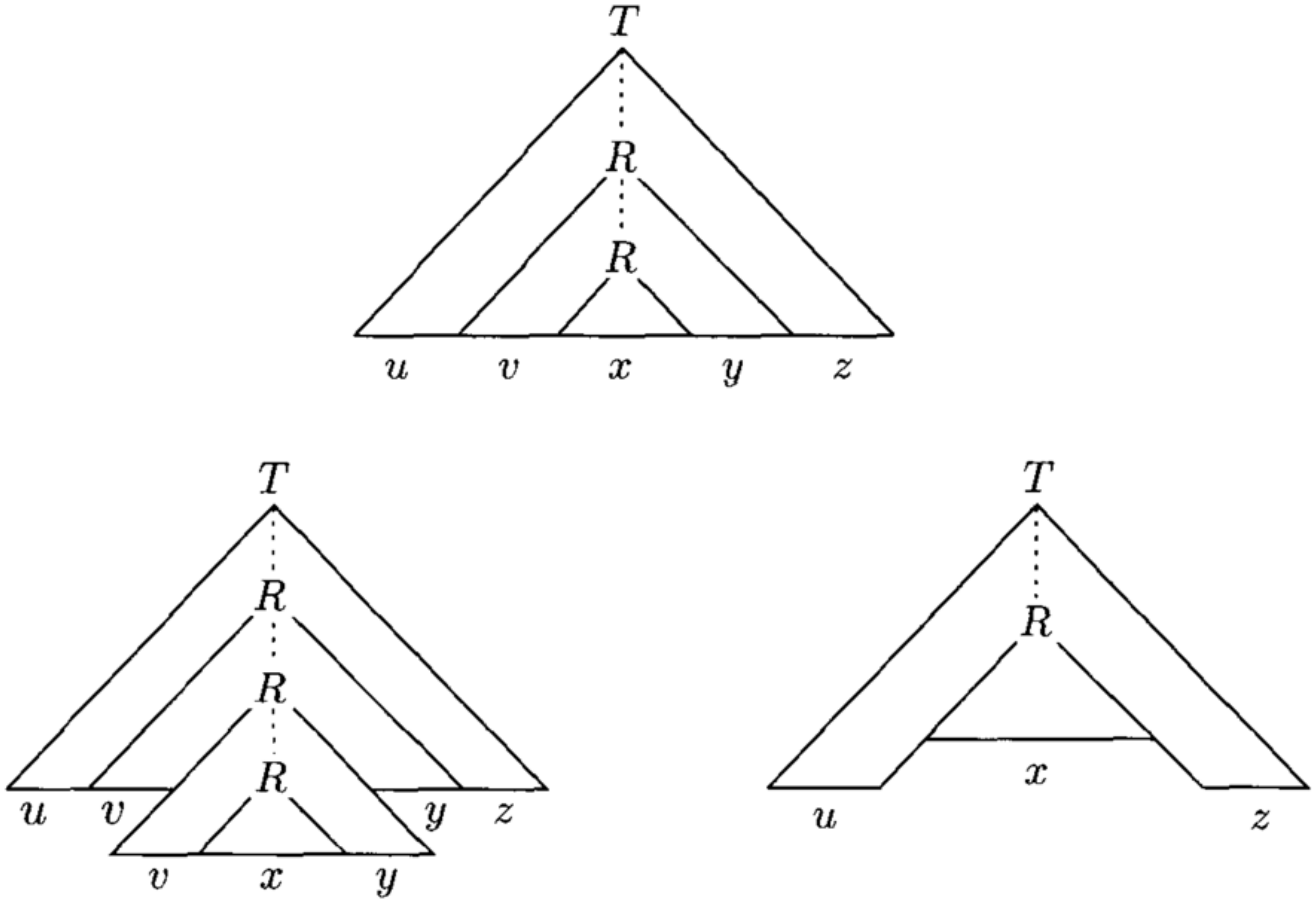


**FIGURE 2.35**
Surgery on parse trees

We divide $s$ into $uvxyz$ according to Figure 2.35. Each occurrence of $R$ has a subtree under it, generating a part of the string $s$. The upper occurrence of $R$ has a larger subtree and generates $vxy$, whereas the lower occurrence generates just $x$ with a smaller subtree. Both of these subtrees are generated by the same variable, so we may substitute one for the other and still obtain a valid parse tree. Replacing the smaller by the larger repeatedly gives parse trees for the strings $uv^ixy^iz$ at each $i > 1$. Replacing the larger by the smaller generates the string $uxz$. That establishes condition 1 of the lemma. We now turn to conditions 2 and 3.

# Pumping Lemma

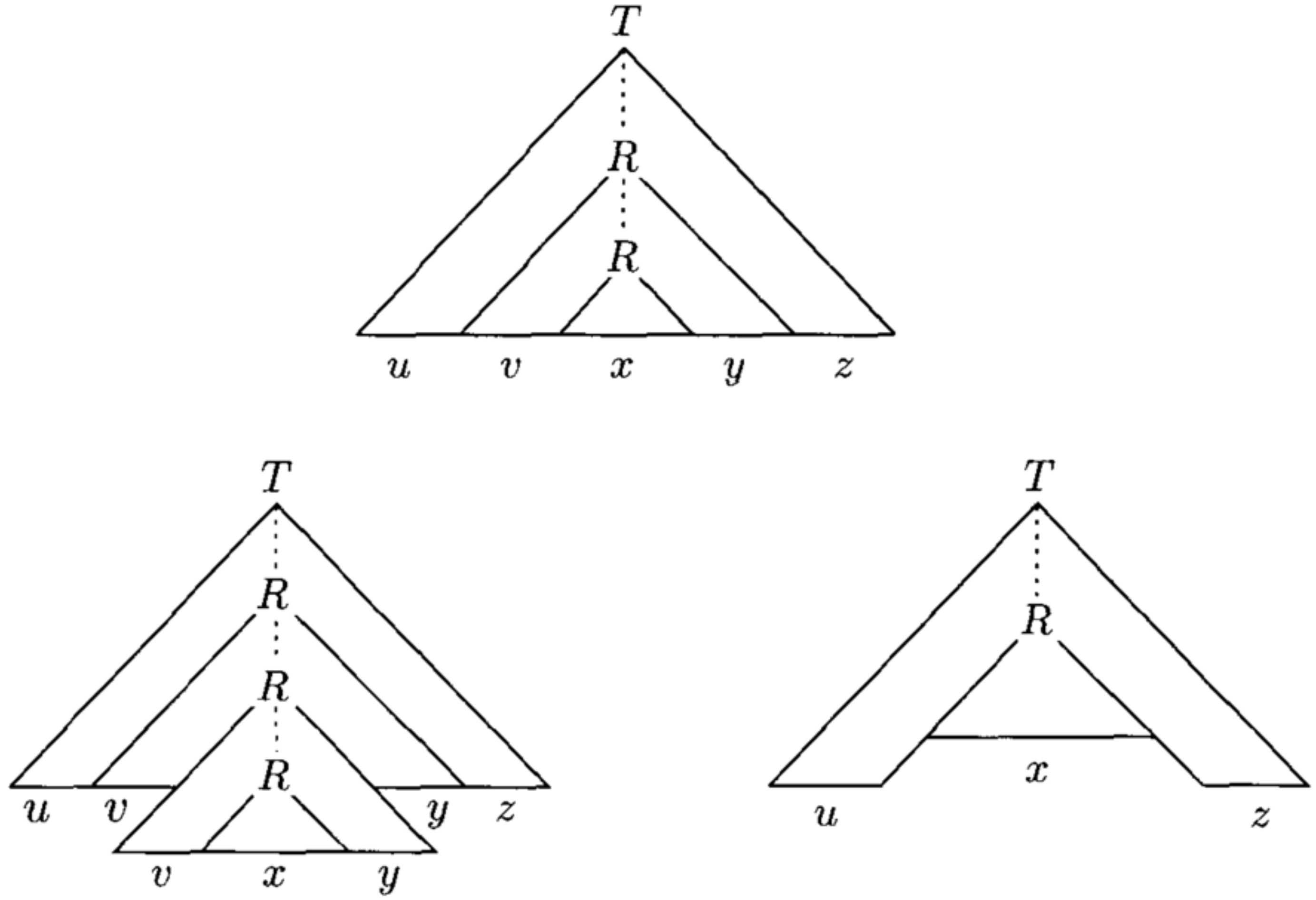


**FIGURE 2.35**
Surgery on parse trees

To get condition 2 we must be sure that both $v$ and $y$ are not $\varepsilon$. If they were, the parse tree obtained by substituting the smaller subtree for the larger would have fewer nodes than $\tau$ does and would still generate $s$. This result isn't possible because we had already chosen $\tau$ to be a parse tree for $s$ with the smallest number of nodes. That is the reason for selecting $\tau$ in this way.

# Pumping Lemma

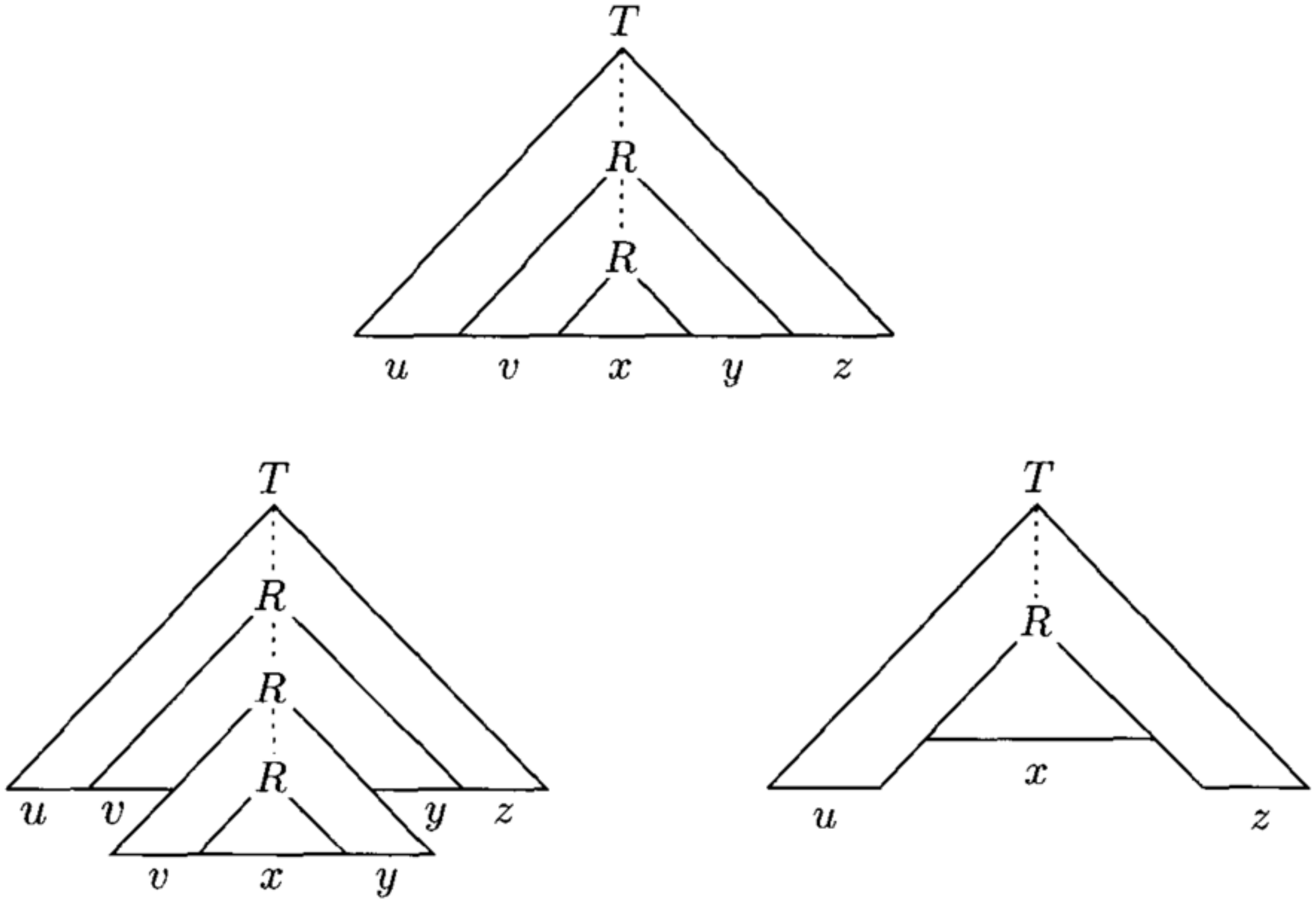


**FIGURE 2.35**
Surgery on parse trees

In order to get condition 3 we need to be sure that $vxy$ has length at most $p$. In the parse tree for $s$ the upper occurrence of $R$ generates $vxy$. We chose $R$ so that both occurrences fall within the bottom $|V| + 1$ variables on the path, and we chose the longest path in the parse tree, so the subtree where $R$ generates $vxy$ is at most $|V| + 1$ high. A tree of this height can generate a string of length at most $b^{|V|+1} = p$.

**THEOREM 2.34**

**Pumping lemma for context-free languages** If $A$ is a context-free language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

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$A \in \mathbb{CFL} \implies$

$\exists p \forall s \in A$, $|s| \geq p$, $\exists uvxyz = s$ st 1,2,3=true.

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$\implies A \notin \mathbb{CFL}$

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$\forall p \exists s \in A$, $|s| \geq p$, $\forall uvxyz = s$ s.t. $|vy| > 0$, $|vxy| < p$,

then $\exists i \geq 0$ s.t. $s' = uv^ixy^iz \notin A$.

$\implies A \notin \mathbb{CFL}$

# NON-CFLs

**EXAMPLE 2.36**

Use the pumping lemma to show that the language $B = \{a^n b^n c^n | \, n \geq 0\}$ is not context free.

We assume that $B$ is a CFL and obtain a contradiction. Let $p$ be the pumping length for $B$ that is guaranteed to exist by the pumping lemma. Select the string $s = a^p b^p c^p$. Clearly $s$ is a member of $B$ and of length at least $p$. The pumping lemma states that $s$ can be pumped, but we show that it cannot. In other words, we show that no matter how we divide $s$ into $uvxyz$, one of the three conditions of the lemma is violated.

# NON-CFLs

First, condition 2 stipulates that either $v$ or $y$ is nonempty. Then we consider one of two cases, depending on whether substrings $v$ and $y$ contain more than one type of alphabet symbol.

1. When both $v$ and $y$ contain only one type of alphabet symbol, $v$ does not contain both a's and b's or both b's and c's, and the same holds for $y$. In this case the string $uv^2xy^2z$ cannot contain equal numbers of a's, b's, and c's. Therefore it cannot be a member of $B$. That violates condition 1 of the lemma and is thus a contradiction.
2. When either $v$ or $y$ contain more than one type of symbol $uv^2xy^2z$ may contain equal numbers of the three alphabet symbols but not in the correct order. Hence it cannot be a member of $B$ and a contradiction occurs.

# NON-CFLs

One of these cases must occur. Because both cases result in a contradiction, a contradiction is unavoidable. So the assumption that $B$ is a CFL must be false. Thus we have proved that $B$ is not a CFL.

# NON-CFLs

**EXAMPLE 2.37**

Let $C = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$. We use the pumping lemma to show that $C$ is not a CFL. This language is similar to language $B$ in Example 2.36, but proving that it is not context free is a bit more complicated.

Assume that $C$ is a CFL and obtain a contradiction. Let $p$ be the pumping length given by the pumping lemma. We use the string $s = a^p b^p c^p$ that we used earlier, but this time we must "pump down" as well as "pump up." Let $s = uvxyz$ and again consider the two cases that occurred in Example 2.36.

# NON-CFLs

$s = uvxyz$ and again consider the two cases that occurred in Example 2.36.

1. When both $v$ and $y$ contain only one type of alphabet symbol, $v$ does not contain both a's and b's or both b's and c's, and the same holds for $y$. Note that the reasoning used previously in case 1 no longer applies. The reason is that $C$ contains strings with unequal numbers of a's, b's, and c's as long as the numbers are not decreasing. We must analyze the situation more carefully to show that $s$ cannot be pumped. Observe that because $v$ and $y$ contain only one type of alphabet symbol, one of the symbols a, b, or c doesn't appear in $v$ or $y$. We further subdivide this case into three subcases according to which symbol does not appear.

# NON-CFLs

**a.** *The* a*'s do not appear*. Then we try pumping down to obtain the string $uv^0xy^0z = uxz$. That contains the same number of a's as $s$ does, but it contains fewer b's or fewer c's. Therefore it is not a member of $C$, and a contradiction occurs.

**b.** *The* b*'s do not appear.* Then either a's or c's must appear in $v$ or $y$ because both can't be the empty string. If a's appear, the string $uv^2xy^2z$ contains more a's than b's, so it is not in $C$. If c's appear, the string $uv^0xy^0z$ contains more b's than c's, so it is not in $C$. Either way a contradiction occurs.

**c.** *The* c*'s do not appear.* Then the string $uv^2xy^2z$ contains more a's or more b's than c's, so it is not in $C$, and a contradiction occurs.

# NON-CFLs

2. When either $v$ or $y$ contain more than one type of symbol, $uv^2xy^2z$ will not contain the symbols in the correct order. Hence it cannot be a member of $C$, and a contradiction occurs.

Thus we have shown that $s$ cannot be pumped in violation of the pumping lemma and that $C$ is not context free.

# NON-CFLs

**EXAMPLE 2.38**

Let $D = \{ww \mid w \in \{0,1\}^*\}$. Use the pumping lemma to show that $D$ is not a CFL. Assume that $D$ is a CFL and obtain a contradiction. Let $p$ be the pumping length given by the pumping lemma.

This time choosing string $s$ is less obvious. One possibility is the string $0^p10^p1$. It is a member of $D$ and has length greater than $p$, so it appears to be a good candidate. But this string *can* be pumped by dividing it as follows, so it is not adequate for our purposes.

$$\overbrace{\underbrace{000\cdots000}_{u}\;\underbrace{0}_{v}\;\underbrace{1}_{x}}^{0^p1}\;\overbrace{\underbrace{0}_{y}\;\underbrace{000\cdots0001}_{z}}^{0^p1}$$

# NON-CFLs

Let's try another candidate for $s$. Intuitively, the string $0^p1^p0^p1^p$ seems to capture more of the "essence" of the language $D$ than the previous candidate did. In fact, we can show that this string does work, as follows.

We show that the string $s = 0^p1^p0^p1^p$ cannot be pumped. This time we use condition 3 of the pumping lemma to restrict the way that $s$ can be divided. It says that we can pump $s$ by dividing $s = uvxyz$, where $|vxy| \leq p$.

# NON-CFLs

First, we show that the substring $vxy$ must straddle the midpoint of $s$. Otherwise, if the substring occurs only in the first half of $s$, pumping $s$ up to $uv^2xy^2z$ moves a 1 into the first position of the second half, and so it cannot be of the form $ww$. Similarly, if $vxy$ occurs in the second half of $s$, pumping $s$ up to $uv^2xy^2z$ moves a 0 into the last position of the first half, and so it cannot be of the form $ww$.

But if the substring $vxy$ straddles the midpoint of $s$, when we try to pump $s$ down to $uxz$ it has the form $0^p1^i0^j1^p$, where $i$ and $j$ cannot both be $p$. This string is not of the form $ww$. Thus $s$ cannot be pumped, and $D$ is not a CFL.

# Reductions (& Construction tools)

- CFLs are closed under union, concatenation and star. If there exists a CFL C s.t. either $A^*=A'$, $A \cup C=A'$, $A \circ C=A'$ (but not complement nor intersection) or any combinations of these operations then A′ is a CFL as long as A is.
- If A′ is NON-CFL then so is A.

# Reduction example

- Consider languages $D=\{ ww \mid w \in \{a,b\}^* \}$, $E=\{ ww \mid w \in \{a,b\}^* \text{ and } |w|<1000 \}$ and $F=D \setminus E$.

- Since $D=E \cup F$ and E is a CFL (a finite and regular language), and since D is a NON-CFL we conclude that

  $$F= \{ ww \mid w \in \{a,b\}^* \text{ and } |w|>999 \}$$

  is also a NON-CFL.

# Reduction example

[A]**2.18** **a.** Let $C$ be a context-free language and $R$ be a regular language. Prove that the language $C \cap R$ is context free.

**b.** Use part (a) to show that the language $A = \{w |\ w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$ and contains equal numbers of a's, b's, and c's$\}$ is not a CFL.

*2.18* **(a)** Let $C$ be a context-free language and $R$ be a regular language. Let $P$ be the PDA that recognizes $C$, and $D$ be the DFA that recognizes $R$. If $Q$ is the set of states of $P$ and $Q'$ is the set of states of $D$, we construct a PDA $P'$ that recognizes $C \cap R$ with the set of states $Q \times Q'$. $P'$ will do what $P$ does and also keep track of the states of $D$. It accepts a string $w$ if and only if it stops at a state $q \in F_P \times F_D$, where $F_P$ is the set of accept states of $P$ and $F_D$ is the set of accept states of $D$. Since $C \cap R$ is recognized by $P'$, it is context free.

**(b)** Let $R$ be the regular language $\mathtt{a}^*\mathtt{b}^*\mathtt{c}^*$. If $A$ were a CFL then $A \cap R$ would be a CFL by part (a). However, $A \cap R = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n |\ n \geq 0\}$, and Example 2.36 proves that $A \cap R$ is not context free. Thus $A$ is not a CFL.

# COMP-330
# Theory of Computation

Fall 2019 -- Prof. Claude Crépeau

# Lec. 13 :
# Pumping Lemma for CFLs