# Winter 2016
# COMP-250: Introduction to Computer Science

Lecture 12, February 18, 2016

# Master Theorem

(CLRS 4.3)

# Master Theorem

Used for many divide-and-conquer recurrences

$$T(n) = aT(n/b) + f(n) \,,$$

where $a \geq 1$, $b > 1$, and $f(n) > 0$.

$a$ = (constant) number of sub-instances,
$b$ = (constant) size ration of sub-instances,
$f$(n) = time used for dividing and recombining.

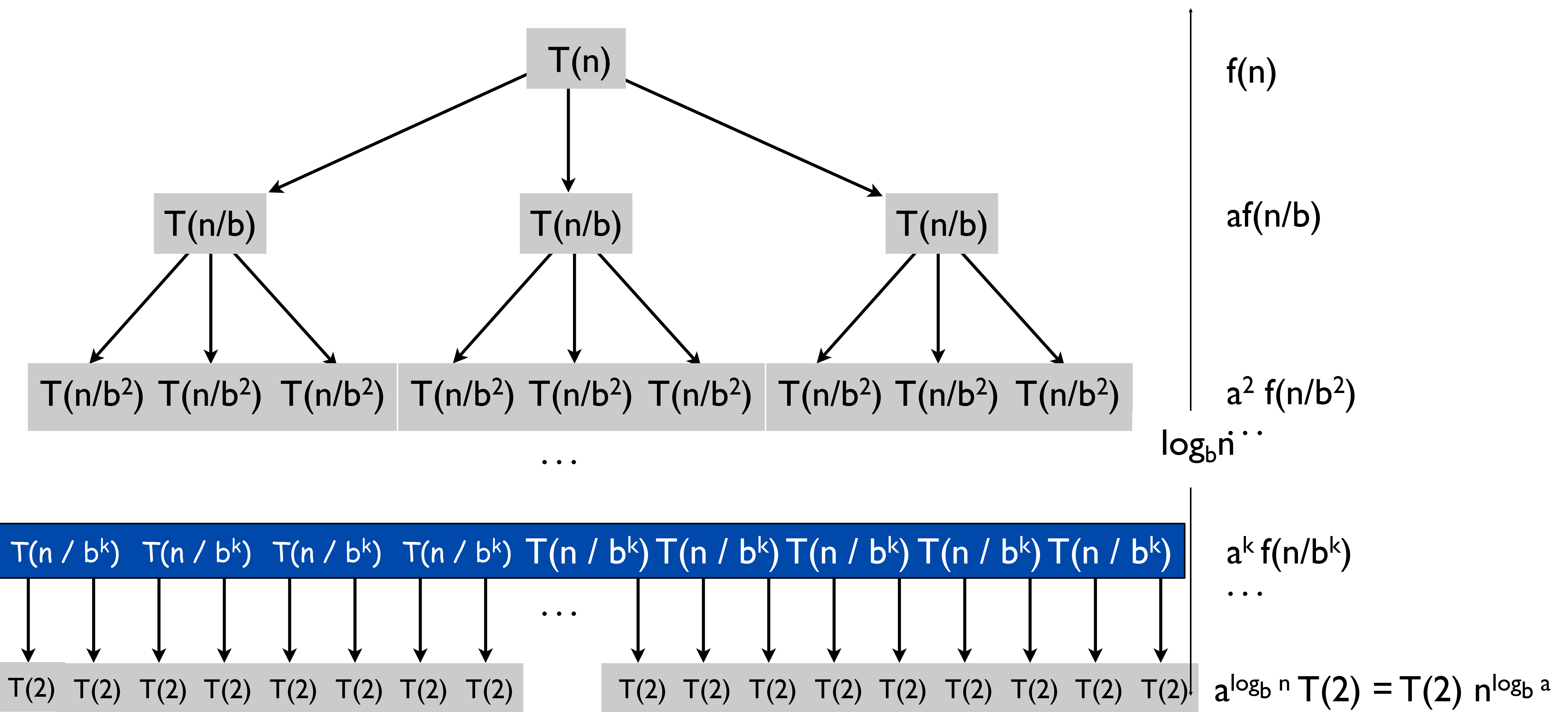
Based on the ***master theorem*** (Theorem 4.1).

Compare $n^{\log_b a}$ vs. $f(n)$:

# Proof by recursion tree

$$T(n) = aT(n/b) + f(n)$$

$T(n) = \sum a^k f(n/b^k)$



# Master Theorem

$$T(n) = aT(n/b) + f(n)$$

**<u>Case 1</u>:** $f(n)$ **is** $O(n^L)$ for some constant $L < \log_b a$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^{\log_b a})$

**<u>Case 2</u>:** $f(n)$ **is** $\Theta(n^{\log_b a} \log^k n)$, for some $k \geq 0$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^{\log_b a} \log^{k+1} n)$

**<u>Case 3</u>:** $f(n)$ **is** $\Omega(n^L)$ for some constant $L > \log_b a$

and $f(n)$ satisfies the regularity condition $af(n/b) \leq cf(n)$ for some $c<1$ and all large $n$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(f(n))$

# Master Theorem

$$T(n) = aT(n/b) + f(n) \,,$$

where $a \geq 1$, $b > 1$, and $f(n) > 0$.

**Case 1:** $f(n)$ **is** $O(n^L)$ for some constant $L < \log_b a$.

($f(n)$ is polynomially smaller than $n^{\log_b a}$.)

**Solution:** $T(n)$ **is** $\Theta(n^{\log_b a})$

(Intuitively: cost is dominated by leaves.)

# Master Theorem

**<u>Case 1</u>:** $f(n)$ **is** $O(n^L)$ for some constant $L < \log_b a$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^{\log_b a})$

$T(n) = 5T(n/2) + \Theta(n^2)$

Compare $n^{\log_2 5}$ vs. $n^2$.

Since $2 < \log_2 5$ use **<u>Case 1</u>**

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^{\log_2 5})$

# Master Theorem

$$T(n) = aT(n/b) + f(n) \ ,$$

where $a \geq 1$, $b > 1$, and $f(n) > 0$.

**Simple Case 2:** $f(n)$ **is** $\Theta(n^{\log_b a})$.

**Solution:** $T(n)$ **is** $\Theta(n^{\log_b a} \log n)$

# Master Theorem

$$T(n) = aT(n/b) + f(n),$$

where $a \geq 1$, $b > 1$, and $f(n) > 0$.

**<u>Case 2</u>:** $f(n)$ **is** $\Theta(n^{\log_b a} \log^k n)$, for some $k \geq 0$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^{\log_b a} \log^{k+1} n)$

(Intuitively: cost is $n^{\log_b a} \lg^k n$ at each level, and there are $\Theta(\lg n)$ levels.)

# Master Theorem

**<u>Case 2</u>:** $f(n)$ **is** $\Theta(n^{\log_b a} \log^k n)$, for some $k \geq 0$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^{\log_b a} \log^{k+1} n)$

$T(n) = 27T(n/3) + \Theta(n^3 \log n)$

Compare $n^{\log_3 27}$ vs. $n^3$.

Since $3 = \log_3 27$ use **<u>Case 2</u>**

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^3 \log^2 n)$

# Master Theorem

$$T(n) = aT(n/b) + f(n) \ ,$$

where $a \geq 1$, $b > 1$, and $f(n) > 0$.

**<u>Case 3</u>:** $f(n)$ **is** $\Omega(n^L)$ for some constant $L > \log_b a$

and $f(n)$ satisfies the regularity condition $af(n/b) \leq cf(n)$ for some $c<1$ and all large $n$.

($f(n)$ is polynomially greater than $n^{\log_b a}$.)

**<u>Solution:</u>** $T(n)$ **is** $\Theta(f(n))$

(Intuitively: cost is dominated by root.)

# Master Theorem

**<u>Case 3</u>:** $f(n)$ **is** $\Omega(n^L)$ for some constant $L > \log_b a$
and $f(n)$ satisfies the regularity condition $af(n/b) \leq cf(n)$ for some $c<1$ and all large $n$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(f(n))$

***What's with the Case 3 regularity condition?***

- Generally not a problem.
- It always holds whenever $f(n) = n^k$ and $f(n)$ **is** $\Omega(n^{\log_b a+\epsilon})$ for constant $\epsilon > 0$.

# Master Theorem

**<u>Case 3</u>:** $f(n)$ **is** $\Omega(n^L)$ for some constant $L > \log_b a$

and $f(n)$ satisfies the regularity condition $af(n/b) \le cf(n)$ for some $c<1$ and all large $n$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(f(n))$

$T(n) = 5T(n/2) + \Theta(n^3)$

Compare $n^{\log_2 5}$ vs. $n^3$.

Since $3 > \log_2 5$ use **<u>Case 3</u>**

$af(n/b) = 5(n/2)^3 = 5/8\ n^3 \le cn^3$, for $c = 5/8$

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^3)$

# ~~Master Theorem~~

**<u>Case 2</u>:** $f(n)$ **is** $\Theta(n^{\log_b a} \log^k n)$, for some $k \geq 0$.

**<u>Solution:</u>** $T(n)$ **is** $\Theta(n^{\log_b a} \log^{k+1} n)$

$T(n) = 27T(n/3) + \Theta(n^3/\log n)$

Compare $n^{\log_3 27}$ vs. $n^3$.

Since $3 = \log_3 27$ use **<u>Case 2</u>**

**<u>but</u>** $n^3/\log n$ **is not** $\Theta(n^3 \log^k n)$ for $k \geq 0$

Cannot use Master Method.

# Divide-and-Conquer Paradigm

# Divide-and-Conquer

**Divide-and-conquer.**
- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

**Most common usage.**
- Break up problem of size n into **two** equal parts of size $^n/_2$.
- Solve two parts recursively.
- Combine two solutions into overall solution in **linear time**.

**Consequence.**
- Straightforward: $n^2$.
- Divide-and-conquer: n log n.

Divide et impera.
Veni, vidi, vici.
- Julius Caesar

# Divide-and-Conquer:
# Binary Search

# Binary Search

**Find a value v in a sorted array of elements.**

$$[a_0 \le a_1 \le, \ldots, \le a_{Size-1}]$$

**Size = number of elements.**

# Binary Search

**Algorithm: binarySearch**$(a, v, low, high)$
**Input:** array $a$, value $v$, lower and upper bound indices $low, high$ ($low = 0, high = n - 1$ initially)
**Output:** the index $i$ of element $v$ (if it is present), -1 (if $v$ is not present)

```
if low == high then
    if a[low] == v then
        return low
    else
        return −1
    end if
else
    mid ← (low + high)/2
    if v ≤ a[mid] then
        return binarySearch(a, v, low, mid )
    else
        return binarySearch(a, v, mid + 1, high)
    end if
end if
```

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 0000 | L | L | | | |
| | 1 | 0001 | | | | | |
| | 2 | 0010 | | | | | |
| | 3 | 0011 | | | | | |
| | 4 | 0100 | | | L | L | |
| * | 5 | 0101 | | | | H | L=H |
| | 6 | 0110 | | | | | |
| | 7 | 0111 | | H | H | | |
| | 8 | 1000 | | | | | |
| | 9 | 1001 | | | | | |
| | 10 | 1010 | | | | | |
| | 11 | 1011 | | | | | |
| | 12 | 1100 | | | | | |
| | 13 | 1101 | | | | | |
| | 14 | 1110 | | | | | |
| | 15 | 1111 | H | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 0000 | L | L | | | |
| | 1 | 0001 | | | | | |
| | 2 | 0010 | | | | | |
| | 3 | 0011 | | | | | |
| | 4 | 0100 | | | L | L | |
| * | 5 | 0101 | | | | H | L=H |
| | 6 | 0110 | | | | | |
| | 7 | 0111 | | H | H | | |
| | 8 | 1000 | | | | | |
| | 9 | 1001 | | | | | |
| | 10 | 1010 | | | | | |
| | 11 | 1011 | | | | | |
| | 12 | 1100 | | | | | |
| | 13 | 1101 | | | | | |
| | 14 | 1110 | | | | | |
| | 15 | 1111 | H | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 0000 | L | L | | | |
| | 1 | 0001 | | | | | |
| | 2 | 0010 | | | | | |
| | 3 | 0011 | | | | | |
| | 4 | 0100 | | | L | L | |
| * | 5 | 0101 | | | | H | L=H |
| | 6 | 0110 | | | | | |
| | 7 | 0111 | | H | H | | |
| | 8 | 1000 | | | | | |
| | 9 | 1001 | | | | | |
| | 10 | 1010 | | | | | |
| | 11 | 1011 | | | | | |
| | 12 | 1100 | | | | | |
| | 13 | 1101 | | | | | |
| | 14 | 1110 | | | | | |
| | 15 | 1111 | H | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 0000 | L | L | | | |
| | 1 | 0001 | | | | | |
| | 2 | 0010 | | | | | |
| | 3 | 0011 | | | | | |
| | 4 | 0100 | | | L | L | |
| * | 5 | 0101 | | | | H | L=H |
| | 6 | 0110 | | | | | |
| | 7 | 0111 | | H | H | | |
| | 8 | 1000 | | | | | |
| | 9 | 1001 | | | | | |
| | 10 | 1010 | | | | | |
| | 11 | 1011 | | | | | |
| | 12 | 1100 | | | | | |
| | 13 | 1101 | | | | | |
| | 14 | 1110 | | | | | |
| | 15 | 1111 | H | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 0000 | L | L | | | |
| | 1 | 0001 | | | | | |
| | 2 | 0010 | | | | | |
| | 3 | 0011 | | | | | |
| | 4 | 0100 | | | L | L | |
| * | 5 | $\boxed{0101}$ | | | | H | L=H |
| | 6 | 0110 | | | | | |
| | 7 | 0111 | | H | H | | |
| | 8 | 1000 | | | | | |
| | 9 | 1001 | | | | | |
| | 10 | 1010 | | | | | |
| | 11 | 1011 | | | | | |
| | 12 | 1100 | | | | | |
| | 13 | 1101 | | | | | |
| | 14 | 1110 | | | | | |
| | 15 | 1111 | H | | | | |

# Binary Search

| | | | | | |
|---|---|---|---|---|---|
| | 0 | L | | | |
| | 1 | | | | |
| | 2 | | | | |
| | 3 | | L | L | |
| * | 4 | | | H | L=H |
| | 5 | H | H | | |

# Binary Search

| | | | | | |
|---|---|---|---|---|---|
| | 0 | L | | | |
| | 1 | | | | |
| | 2 | | | | |
| | 3 | | L | L | |
| * | 4 | | | H | L=H |
| | 5 | H | H | | |

# Binary Search

```
  0   L
  1
  2
  3       L   L
* 4           H  L=H
  5   H   H
```

# Binary Search

| | | | | |
|---|---|---|---|---|
| 0 | L | | | |
| 1 | | | | |
| 2 | | | | |
| 3 | | L | L | |
| * 4 | | | H | L=H |
| 5 | H | H | | |

# Recurrence Relation

**Def.** T(n) = number of comparisons to find v among n sorted elements.

**Binary Search recurrence.**

$$
T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n > 1 \end{cases}
$$

**Solution.** T(n) is O(log n) (Master Theorem Case 2).

# Divide-and-Conquer Multiplication

# Integer Multiplication

**Add.** Given two n-digit integers a and b, compute a + b.

- $\Theta(n)$ bit operations.

**Multiply.** Given two n-digit integers a and b, compute a × b.

- Grade School solution: $\Theta(n^2)$ bit operations.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | | | | | | | | × | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| | | | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| | | | | | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| | | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | | |
| | | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | | | |
| | | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | | | | |
| | | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | | | | | |
| | | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | | | | | | |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

Multiply

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| + | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |

Add

# D&C Multiplication

**To multiply two n-digit integers:**

- Multiply four $^n/_2$-digit integers.
- Add two $^n/_2$-digit integers, and shift to obtain result.

$$x = 2^{n/2} \cdot x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) = 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0$$

$$\mathrm{T}(n) = \underbrace{4T\left(n/2\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow \mathrm{T}(n) \text{ is } \Theta(n^2)$$

assumes n is a power of 2

# Telescoping Proof

**Claim.**

$$T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) \text{ is } \Theta(n^2)$$

assumes n is a power of 2

**Pf.** For n > 1:

$$\begin{aligned}
T(n)/n &= 4T(n/2)/n + C \\
&= 2\,T(n/2)/(n/2) + C \\
&= 2\,[\,2\,T(n/4)/(n/4) + C\,] + C \\
&= 4\,T(n/4)/(n/4) + 2C + C \\
&= 4\,[\,2\,T(n/8)/(n/8) + C\,] + 2C + C \\
&= 8\,T(n/8)/(n/8) + 4C + 2C + C \\
&\ldots \\
&= n\,T(1)/1 + n/2\,C + n/4\,C + \ldots + 4C + 2C + C \\
&= C\,(n/2+n/4+\ldots+2+1) = C(n-1).
\end{aligned}$$

# Karatsuba Multiplication

**To multiply two n-digit integers:**

- Add two $^n/_2$ digit integers.
- Multiply **three** $^n/_2$-digit integers.
- Add, subtract, and shift $^n/_2$-digit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
&= 2^n \cdot \underset{A}{x_1 y_1} + 2^{n/2} \cdot \left( \underset{B}{(x_1 + x_0)(y_1 + y_0)} - \underset{A}{x_1 y_1} - \underset{C}{x_0 y_0} \right) + \underset{C}{x_0 y_0}
\end{aligned}
$$

**Theorem.** [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$
\mathrm{T}(n) \le \underbrace{T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + T\left(1 + \lceil n/2 \rceil\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}
$$

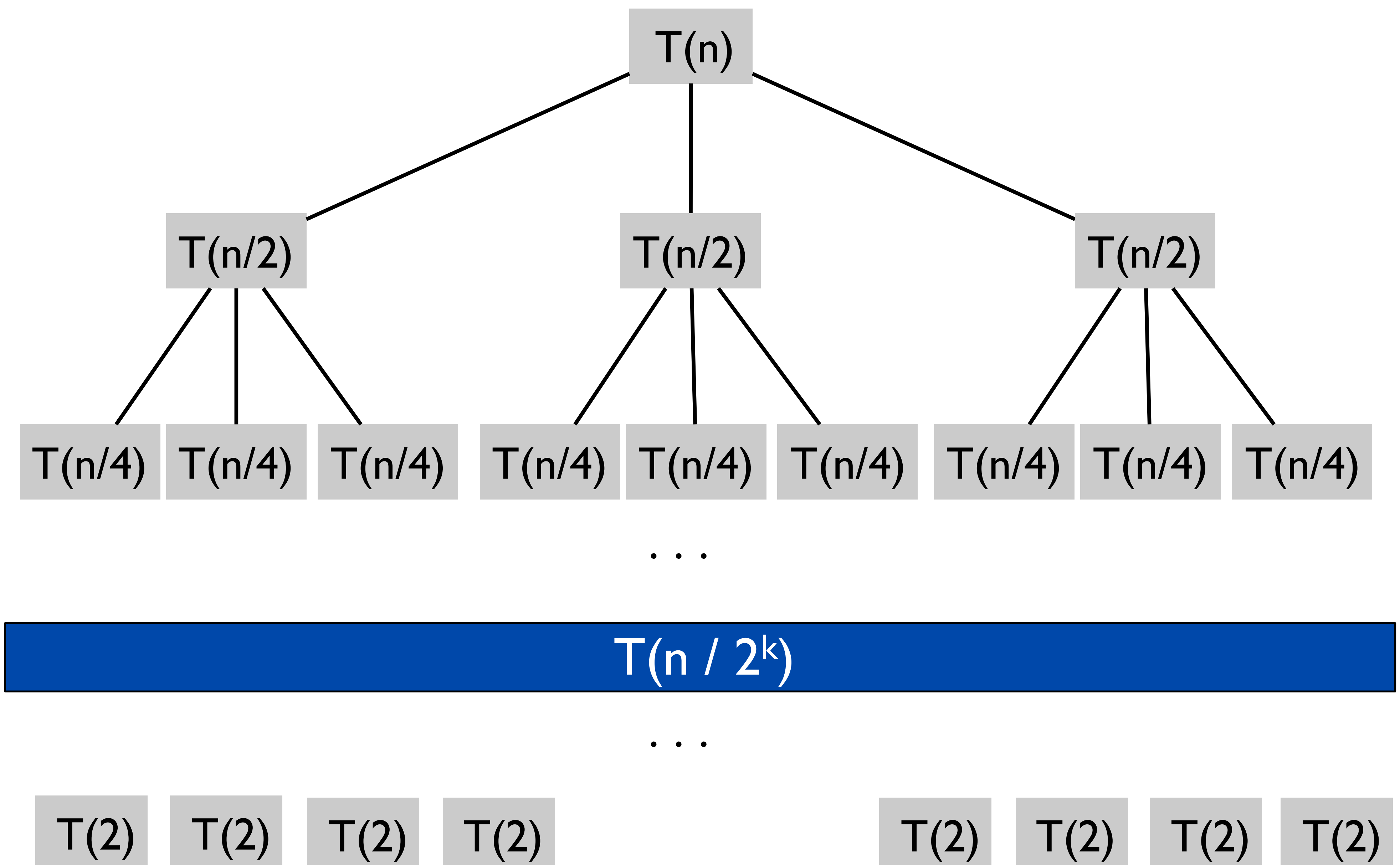
$$
\Rightarrow \mathrm{T}(n) \text{ is } O(n^{\log_2 3}) \text{ is } O(n^{1.585})
$$

$$\sum_{k=0}^{n-1} ar^k = a\,\frac{1-r^n}{1-r}$$

# Karatsuba Recursion Tree

$$\mathrm{T}(n) = \begin{cases} 0 & \text{if } n = 1 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$\mathrm{T}(n) = \sum_{k=0}^{\log_2 n} n \left(\tfrac{3}{2}\right)^k = n\frac{\left(\frac{3}{2}\right)^{1+\log_2 n} - 1}{\frac{3}{2} - 1} = 3n^{\log_2 3} - 2$$




n

3(n/2)

9(n/4)

. . .

$3^k\,(n / 2^k)$

. . .

$3^{\lg n}\,(2)$

# Karatsuba Multiplication

**Generalization:** $O(n^{1+\varepsilon})$ for any $\varepsilon > 0$.

**Best known:** $n \log n \, 2^{O(\log^* n)}$

$$\text{where } \log^*(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 + \log^*(\log x) & \text{if } x > 1 \end{cases}$$

**Conjecture:** $\Omega(n \log n)$ but not proven yet.

# Winter 2016
# COMP-250: Introduction to Computer Science

Lecture 12, February 18, 2016