Winter 2016
COMP-250: Introduction to Computer Science

Lecture 10, February 11, 2016

A Survey of Common Running Times

Linear Time: O(n)

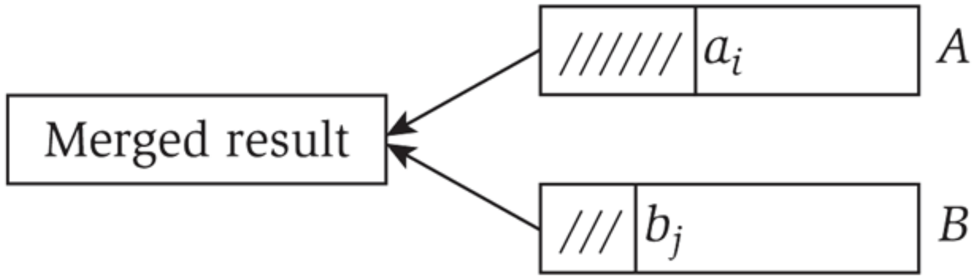
Linear time. Running time is proportional to input size.

Computing the maximum. Compute maximum of n numbers $a_1, \ldots, a_n$.

```
max ← $a_1$
for i = 2 to n {
   if ($a_i$ > max)
      max ← $a_i$
}
```

Linear Time: O(n)

Merge. Combine two sorted lists **$A = a_1,a_2,\ldots,a_n$ with $B = b_1,b_2,\ldots,b_n$** into a sorted whole.



```
i = 1, j = 1
while (both lists are nonempty) {
   if ($a_i \le b_j$) append $a_i$ to output list and increment i
   else       append $b_j$ to output list and increment j
}
append remainder of nonempty list to output list
```

Claim. Merging two lists of size n takes O(n) time.

Pf. After each comparison, the length of output list increases by 1.

O(n log n) Time

O(n log n) time. Arises in divide-and-conquer algorithms.

also referred to as linearithmic time

Sorting. **Mergesort** and **Heapsort** are sorting algorithms that perform O(n log n) comparisons.

Largest empty interval. Given n time-stamps $x_1, \ldots, x_n$ on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

O(n log n) solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

Quadratic Time: $O(n^2)$

Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

```
min ← $(x_1 - x_2)^2 + (y_1 - y_2)^2$
for i = 1 to n {
   for j = i+1 to n {
      d ← $(x_i - x_j)^2 + (y_i - y_j)^2$
      if (d < min)
         min ← d
   }
}
```

don't need to
take square roots

Remark. This algorithm is $\Omega(n^2)$ and it seems inevitable in general, but this is just an illusion.

Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Given n sets $S_1, \ldots, S_n$ each of which is a subset of 1, 2, …, n, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pair of sets, determine if they are disjoint.

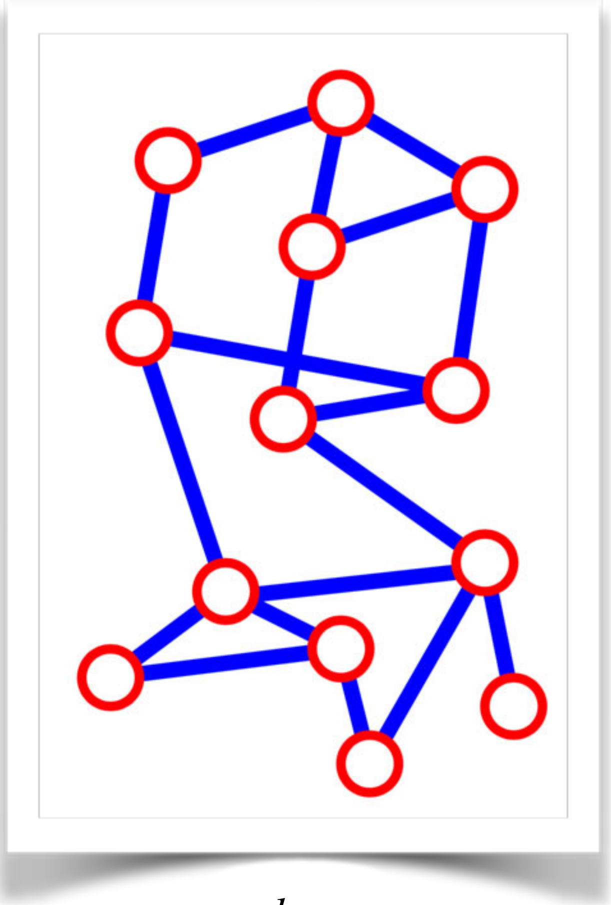
```
foreach set $S_i$ {
   foreach other set $S_j$ {
      foreach element p of $S_i$ {
         determine whether p also belongs to $S_j$
      }
      if (no element of $S_i$ belongs to $S_j$)
         report that $S_i$ and $S_j$ are disjoint
   }
}
```

Polynomial Time: $O(n^k)$

Independent set of size k. Given a graph, are there k nodes such that no two are joined by an edge?

k is a constant

$O(n^k)$ solution. Enumerate all subsets of k nodes.



```
foreach subset S of k nodes {
   check whether S in an independent set
   if (S is an independent set)
      report S is an independent set
   }
}
```

- Check whether S is an independent set = $O(k^2)$.
- Number of k element subsets : $\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\,\cdots\,(n-k+1)}{k\,(k-1)\,(k-2)\,\cdots\,(2)\,(1)} \leq \frac{n^k}{k!}$
- $O(k^2\, n^k / k!)$ is $O(n^k)$.

poly-time for k=17,
but not practical

Polynomial Time: O(n^k)

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O(n^k) solution. Enumerate all subsets of k nodes.

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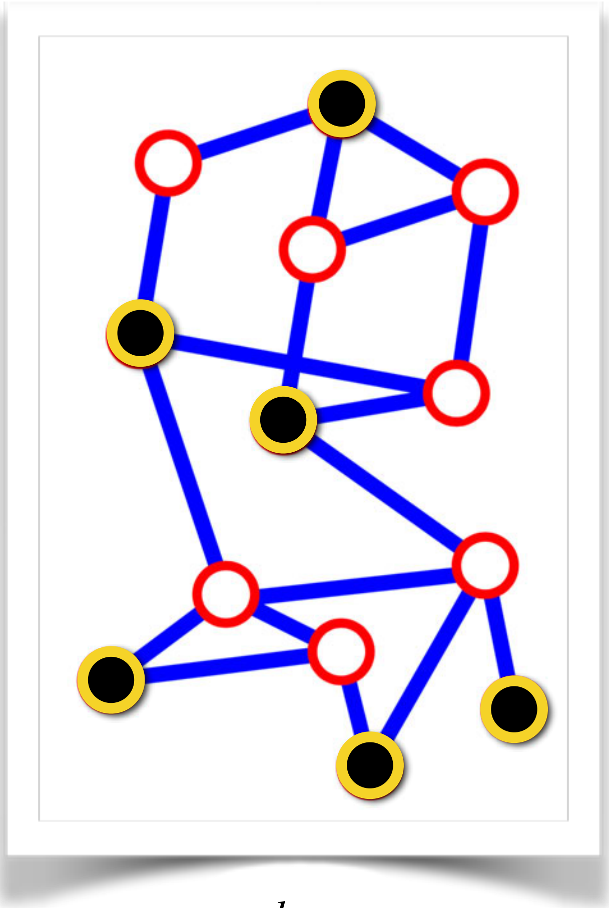

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- O(k^2 n^k / k!) is O(n^k).

poly-time for k=17,
but not practical

Exponential Time: $O(c^n)$

Independent set. Given a graph, what is the maximum size of an independent set?

$O(n^2\, 2^n)$ solution. Enumerate all subsets.

```
S* ← ∅
foreach subset S of nodes {
   check whether S in an independent set
   if (S is largest independent set seen so far)
      update S* ← S
   }
}
```

Induction and Recursion

Induction Proofs

Predicate.

- $P(n)$: $f(n)$ = some formula in n

Statement.

$\forall n \geq 1$, $P(n)$ is true.

Proof.

- Base case: proof that $P(1)$ is true.
- Induction step: $\forall n \geq 1$, $P(n) \Longrightarrow P(n+1)$.

Let $n \geq 1$.

Assume for induction hypothesis that $P(n)$ is true and prove $P(n+1)$ is also true.

Induction Proof (1)

Predicate.

- P(n) : 1+2+…+n = n(n+1)/2

- Base case: when n=1 we have
  1+…+n = 1 = 1(2)/2 = n(n+1)/2.
  P(1) is true.

- Induction step: let n≥1. Assume for induction hypothesis that P(n) is true.
  We show P(n+1) is true as well :
  1+2+…+n+(n+1) = n(n+1)/2 + (n+1) by I.H.
  = (n+1)(n/2 + 1)
  = (n+1)(n+2)/2.
  n≥1, P(n) ⟹ P(n+1).

Induction Proof (II)

Predicate.

- $P(n) : \sum_{i=1}^{n} i = n(n+1)/2$
- Base case: when $n=1$, $\sum_{i=1}^{1} i = 1 = 1(2)/2 = n(n+1)/2$.

  P(1) is true.
- Induction step: let $n \geq 1$. Assume for induction hypothesis that P(n) is true.
  We show P(n+1) is true as well :

  $$\sum_{i=1}^{n+1} i = (n+1) + \sum_{i=1}^{n} i$$

  $$= (n+1) + n(n+1)/2 \quad \text{by I.H.}$$
  $$= (n+1)(1+n/2)$$
  $$= (n+1)(n+2)/2.$$

  $n \geq 1, P(n) \Rightarrow P(n+1)$.

Iteration vs Recursion

- $f(n) = 1 + 2 + \ldots + n = \sum_{i=1}^{n} i$

```
f(n)
sum ← 0
for i = 2 to n {
    sum ← sum + i
}
return sum
```

- $f(n) = \begin{cases} 0 & \text{if } n = 0 \\ f(n-1)+n & \text{if } n > 0 \end{cases}$

```
f(n)
if n = 0 { return 0 }
else { return f(n-1)+n }
```

Induction Proof (III)

Predicate.

- $P(n)$: $f(n) = n(n+1)/2$

- Base case: when $n=1$, $f(1) = 1 = 1(2)/2 = n(n+1)/2$.

  $P(1)$ is true.

- Induction step: let $n \geq 1$. Assume for induction hypothesis that $P(n)$ is true.
  We show $P(n+1)$ is true as well :

  $$\begin{aligned} f(n+1) &= f(n) + (n+1) && \text{by definition} \\ &= n(n+1)/2 + (n+1) && \text{by I.H.} \\ &= (n+1)(n/2+1) \\ &= (n+1)(n+2)/2. \end{aligned}$$

  $n \geq 1, P(n) \Longrightarrow P(n+1)$.

Generalized Induction Proofs

Predicate.

- P(n) : f(n) = some formula in n

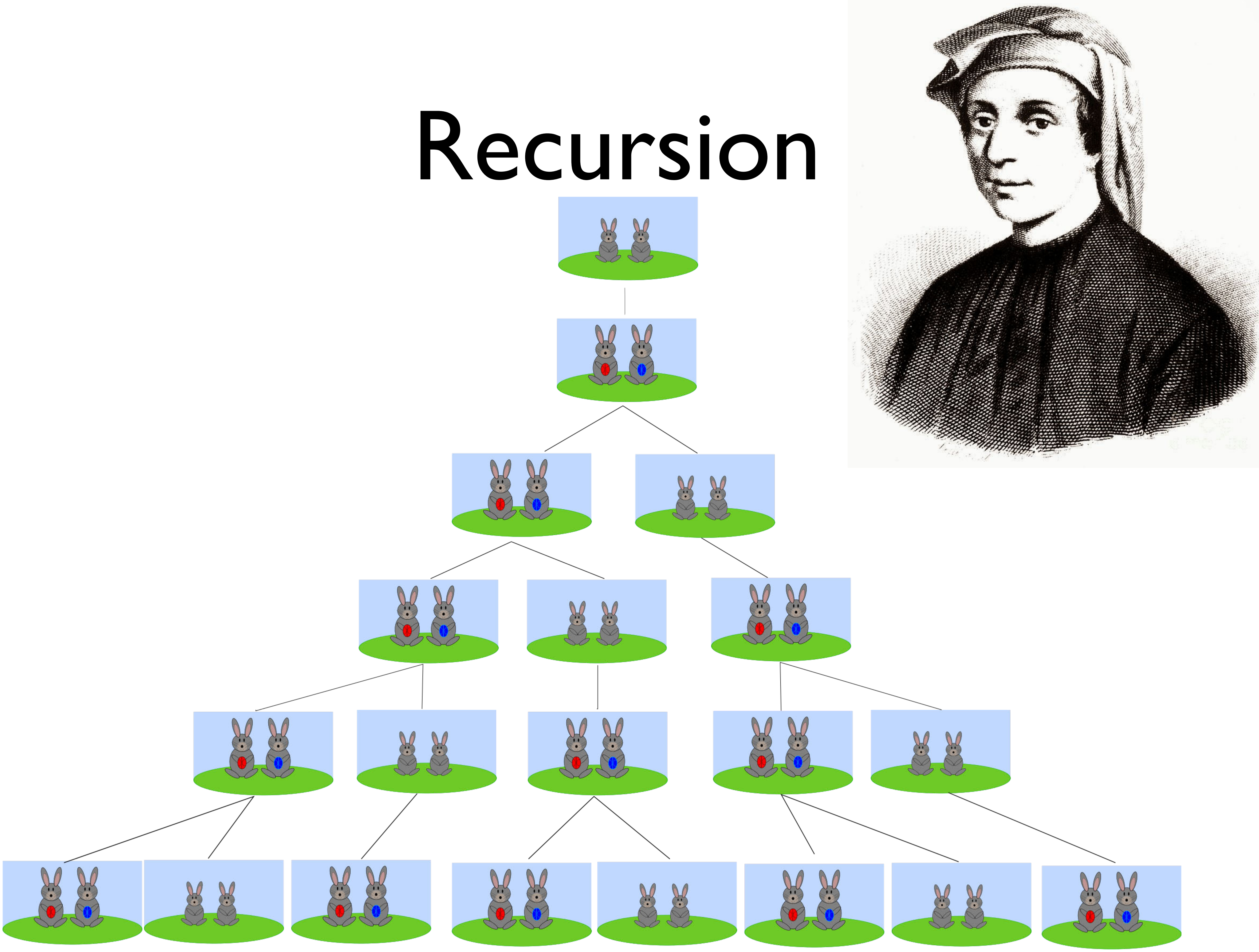
Statement.

For all n≥1, P(n) is true.

Proof.

- Base case: proof that P(1) is true.
- Induction step: let n≥1. Assume for induction hypothesis that P(1)…P(n) are all true. We show P(n+1) is also true.

Recursion



Recursion:
Fibonacci Sequence

- $$\text{fib}(n) = \begin{cases} n & \text{if } n \leq 1 \\ \text{fib}(n-1) + \text{fib}(n-2) & \text{if } n > 1 \end{cases}$$

  Fibonacci sequence:
  0,1,1,2,3,5,8,13,21,34,55,89,144,…

- NOT so easy to define iteratively…

Recursion vs Iteration

- $\text{fib}(n) = \begin{cases} n & \text{if } n \leq 1 \\ \text{fib}(n-1) + \text{fib}(n-2) & \text{if } n > 1 \end{cases}$

```
fib(n)
if n < 2 { return n }
else { return fib(n-1)+ fib(n-2) }
```

```
fib(n)
a ← 0
b ← 1
for i = 1 to n {
   b ← a + b
   a ← b - a
}
return a
```

Generalized Induction Proofs

Statement.
For all $n \geq 0$, $P(n)$: $\mathrm{fib}(n) \leq 2^n$ is true.

Proof.

- Base case: $P(0)$: $\mathrm{fib}(0) = 0 \leq 2^0$ is true.
  $P(1)$: $\mathrm{fib}(1) = 1 \leq 2^1$ is true.

- Induction step: let $n \geq 1$. Assume for induction hypothesis that $P(0)\ldots P(n)$ are all true. We show $P(n+1)$ is also true:

$$\begin{aligned} \mathrm{fib}(n+1) &= \mathrm{fib}(n) + \mathrm{fib}(n-1) && \text{by definition} \\ &\leq 2^n + 2^{n-1} && \text{by gen. I. H.} \\ &\leq 2^{n-1} \bullet 3 < 2^{n+1} \end{aligned}$$

Generalized Induction Proofs

Statement.

For all $n \geq 1$, $P(n)$: $\text{fib}(n) \leq \varphi^n$ is true.

Proof.

- Base case: $P(1)$: $\text{fib}(1) = 1 \leq \varphi^1$ is true (if $\varphi \geq 1$).
  $P(2)$: $\text{fib}(2) = 1 \leq \varphi^2$ is true (if $\varphi \geq 1$).

- Induction step: let $n \geq 1$. Assume for induction hypothesis that $P(1)\ldots P(n)$ are all true. We show $P(n+1)$ is also true:

  $\text{fib}(n+1) = \text{fib}(n) + \text{fib}(n-1)$ by definition
  $\leq \varphi^n + \varphi^{n-1}$ by gen. I. H.
  $\leq \varphi^{n-1}(\varphi+1) \leq \varphi^{n+1}$
  whenever $(\varphi+1) \leq \varphi^2$
  whenever $0 \leq \varphi^2 - \varphi - 1$.

Generalized Induction Proofs

Statement.

For all $n \geq 1$, $P(n)$: $\mathrm{fib}(n) \geq \varphi^{n-2}$ is true.

Proof.

- Base case: $P(1)$: $\mathrm{fib}(1) = 1 \geq \varphi^{-1}$ is true (if $\varphi \geq 1$).
  $P(2)$: $\mathrm{fib}(2) = 1 = \varphi^{0}$ is true.

- Induction step: let $n \geq 1$. Assume for induction hypothesis that $P(1) \ldots P(n)$ are all true. We show $P(n+1)$ is also true:

  $\mathrm{fib}(n+1) = \mathrm{fib}(n) + \mathrm{fib}(n-1)$ by definition
  $\geq \varphi^{n-2} + \varphi^{n-3}$ by gen. I. H.
  $\geq \varphi^{n-3} (\varphi+1) \geq \varphi^{n-1}$
  whenever $(\varphi+1) \geq \varphi^{2}$
  whenever $0 \geq \varphi^{2}-\varphi-1$.

Weak Binet Formula

Statements.
For all $n \geq 1$, $\mathrm{fib}(n) \leq \varphi^{n}$ is true.
whenever $0 \leq \varphi^{2}-\varphi-1$ and $\varphi \geq 1$.

For all $n \geq 1$, $\mathrm{fib}(n) \geq \varphi^{n-2}$ is true.
whenever $0 \geq \varphi^{2}-\varphi-1$ and $\varphi \geq 1$.

Therefore:
For all $n \geq 1$, $\varphi^{n}/\varphi^{2} \leq \mathrm{fib}(n) \leq \varphi^{n}$ is true.
whenever $0 = \varphi^{2}-\varphi-1$ and $\varphi \geq 1$.
Only solution φ = golden ratio = $(1+\sqrt{5})/2$.

fib(n) is $\boldsymbol{\theta}(\varphi^{n})$.

Generalized Induction Proofs

- $$f(n)=\begin{cases} n & \text{if } n \leq 1 \\ f^2({}^{n+1}/_2) + f^2({}^{n-1}/_2) & \text{if odd } n>1 \\ f^2({}^{n}/_2+1) - f^2({}^{n}/_2-1) & \text{if even } n>1 \end{cases}$$

  f-sequence:

  0,1,1,2,3,5,8,13,21,34,55,89,144,…

Statement.

For all $n \geq 0$, $\mathrm{fib}(n) = f(n)$.

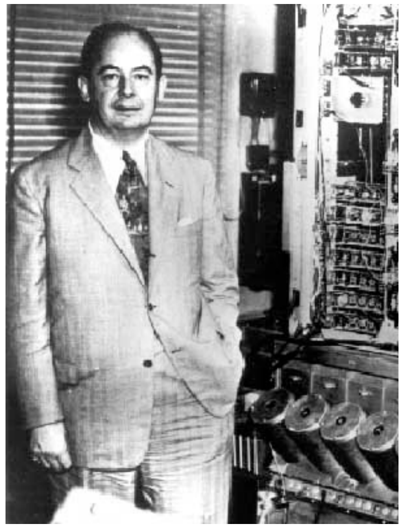
Left as an exercise…

Recursive Algorithms

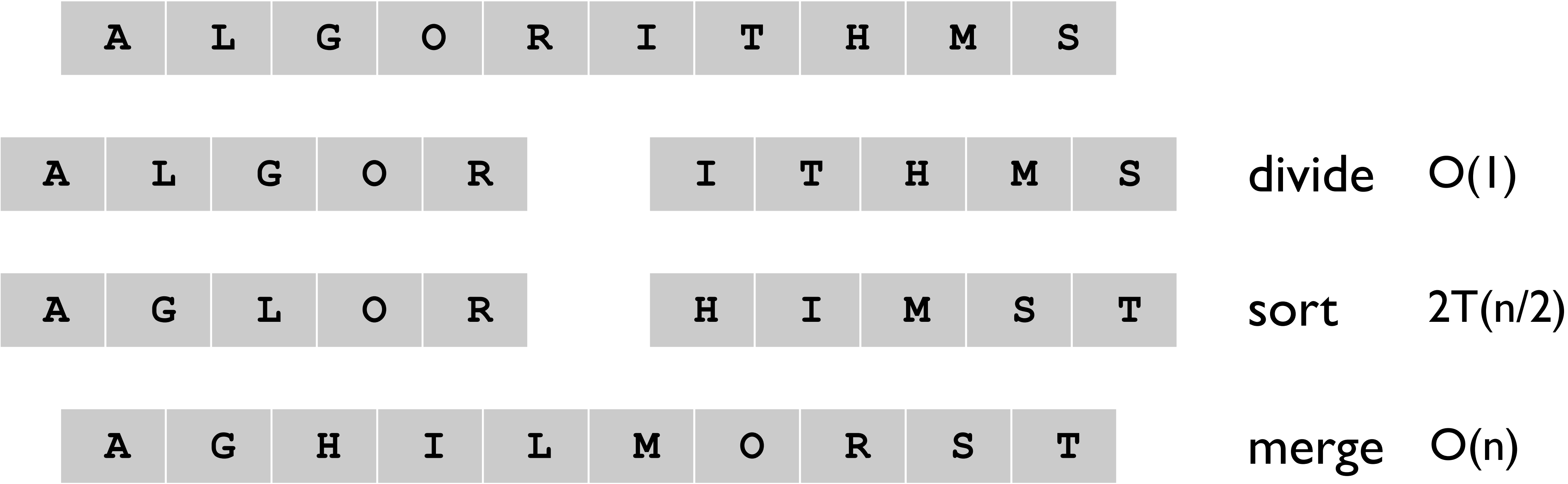
Merge Sort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.



Jon von Neumann
(1945)



A L G O R I T H M S

A L G O R   I T H M S   divide O(1)

A G L O R   H I M S T   sort 2T(n/2)

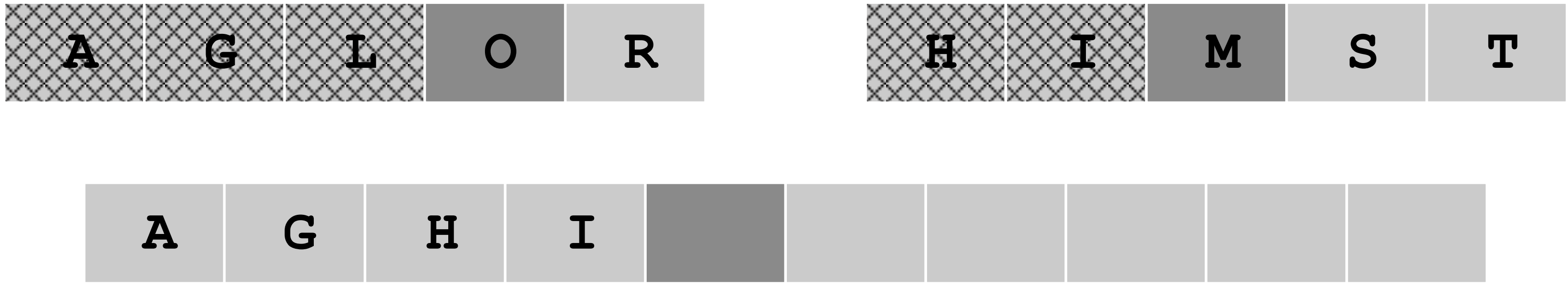
A G H I L M O R S T   merge O(n)

Merge

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently?

- Linear number of comparisons.
- Use temporary array.



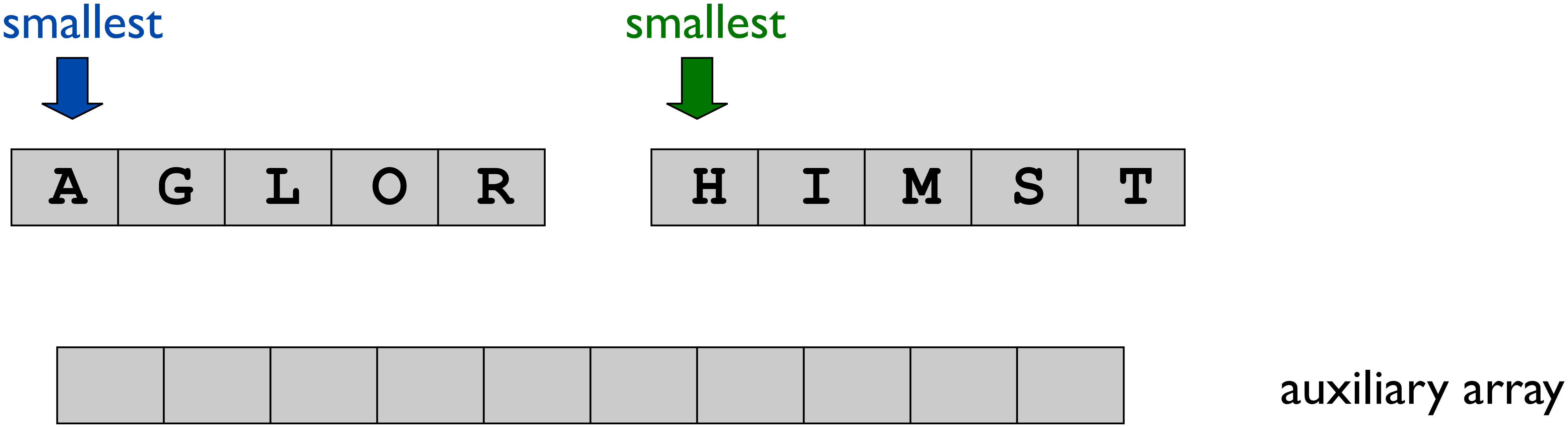
Challenge for the bored. In-place merge. [Kronrod, 1969]

using only a constant amount of extra storage

Merge

Merging.

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into auxiliary array.
- Repeat until done.



smallest

smallest

A G L O R

H I M S T

auxiliary array

Merge

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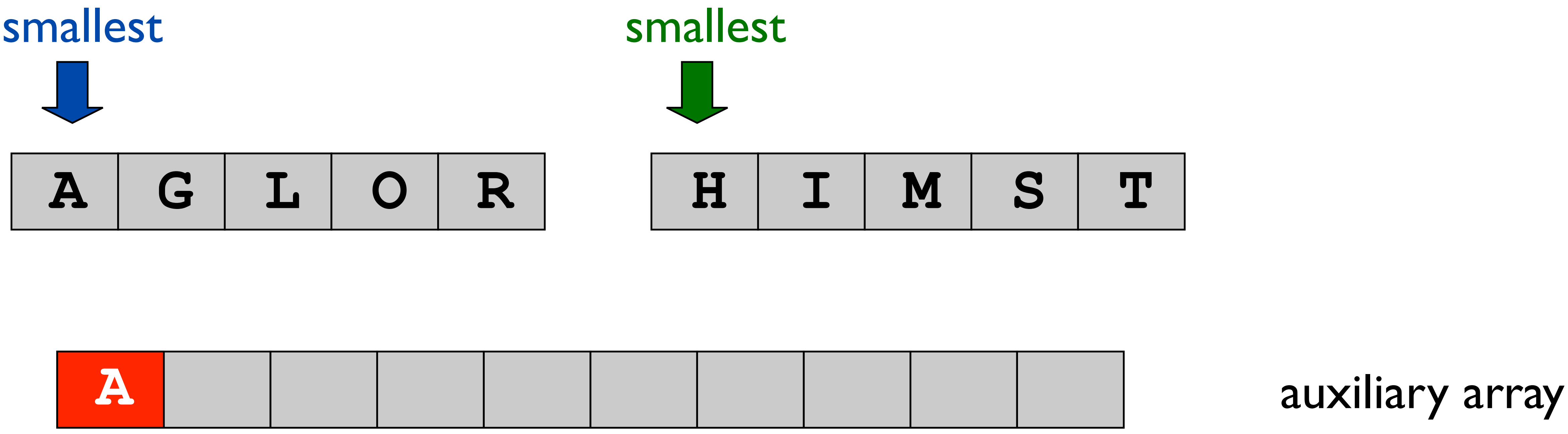

smallest

smallest

A	G	L	O	R

H	I	M	S	T

A									

auxiliary array

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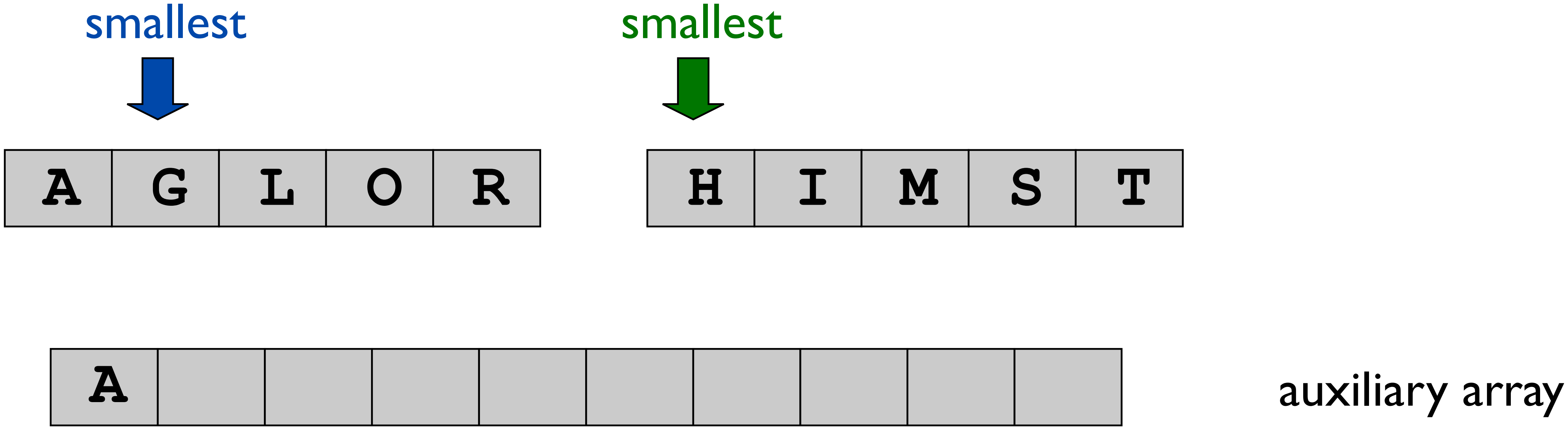

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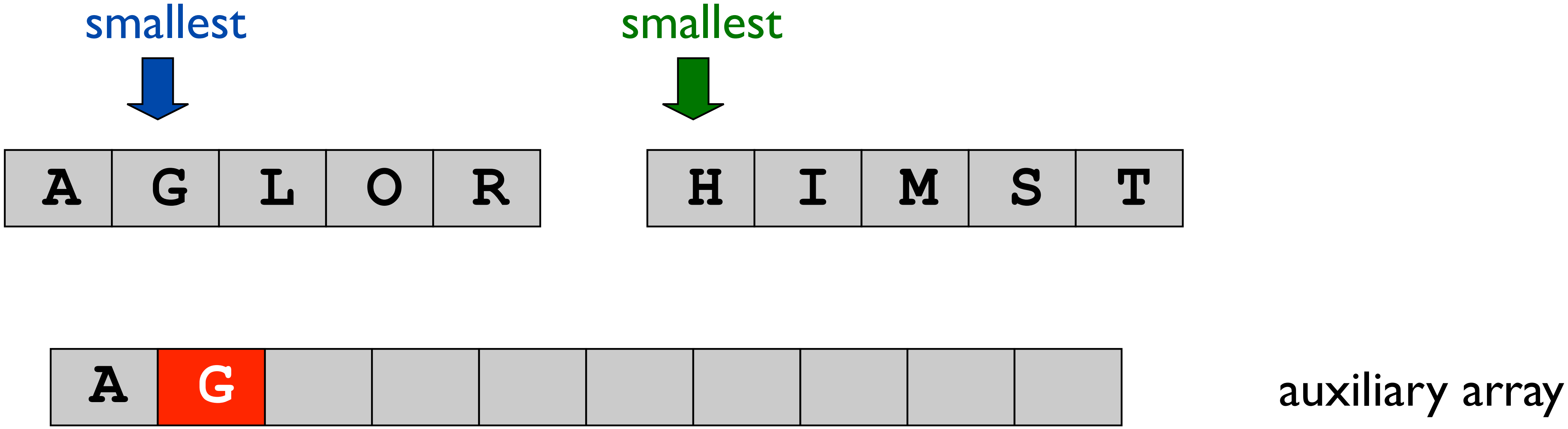

smallest

smallest

A G L O R

H I M S T

A G

auxiliary array

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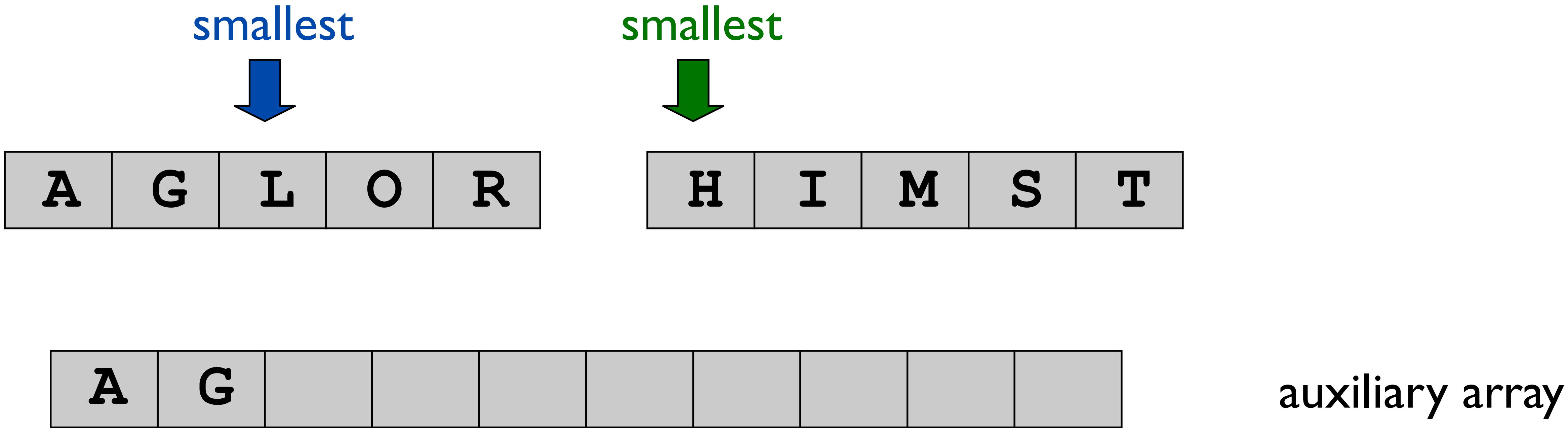

smallest smallest

A G L O R H I M S T

A G

auxiliary array

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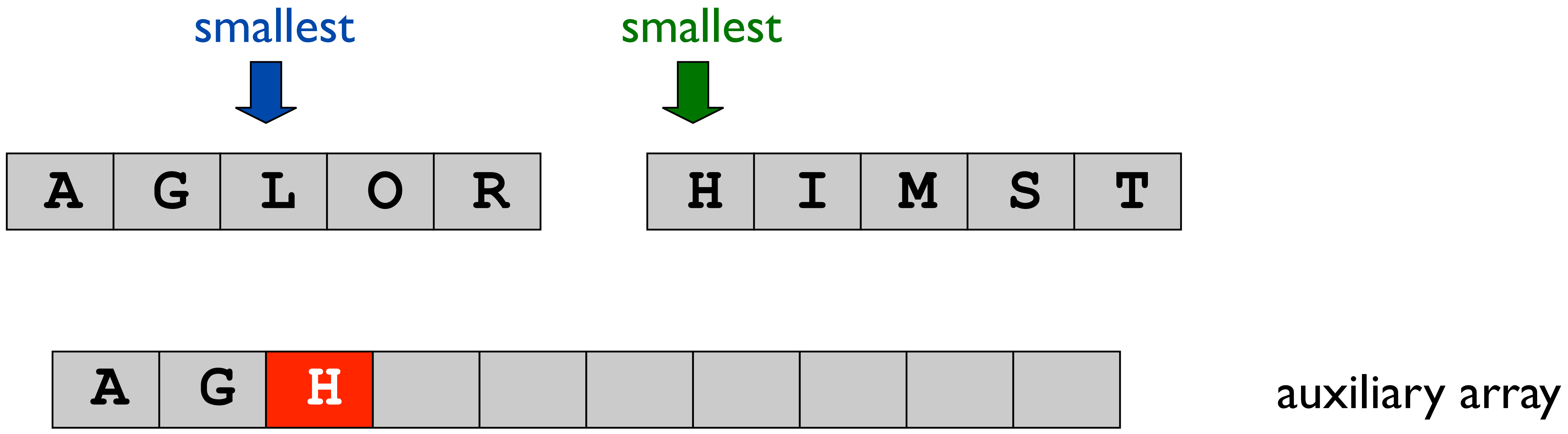

smallest

smallest

A	G	L	O	R

H	I	M	S	T

A	G	H							

auxiliary array

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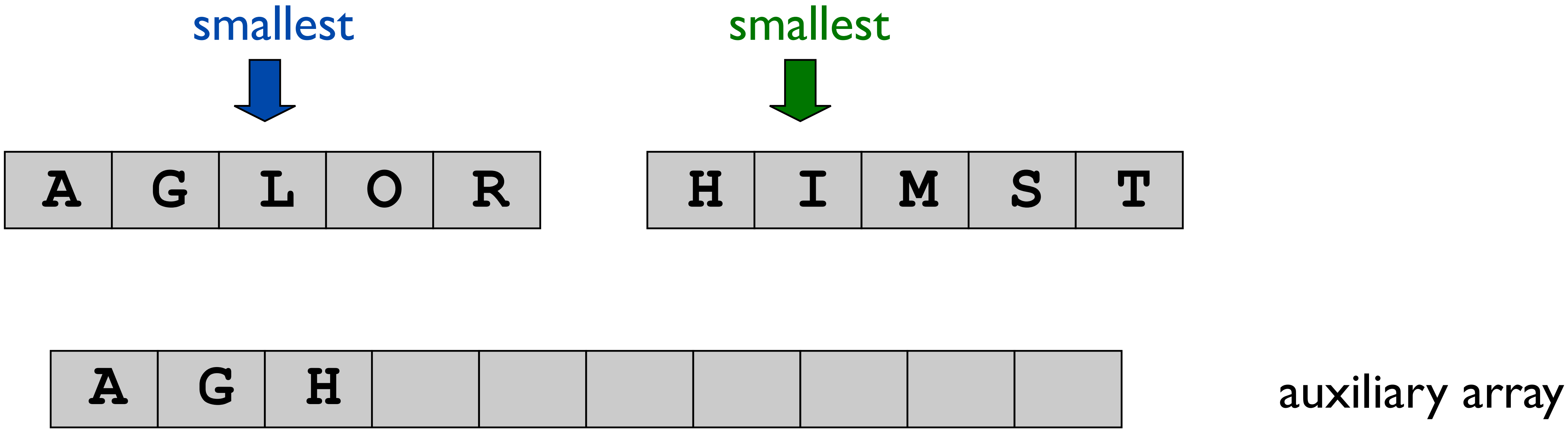

smallest

smallest

A	G	L	O	R

H	I	M	S	T

A	G	H							

auxiliary array

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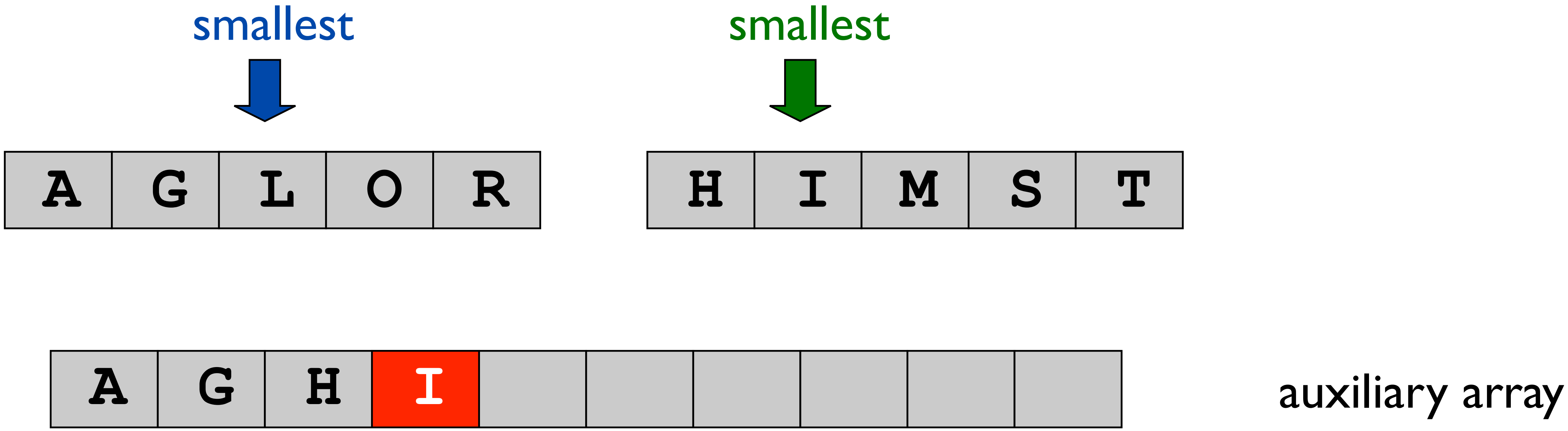

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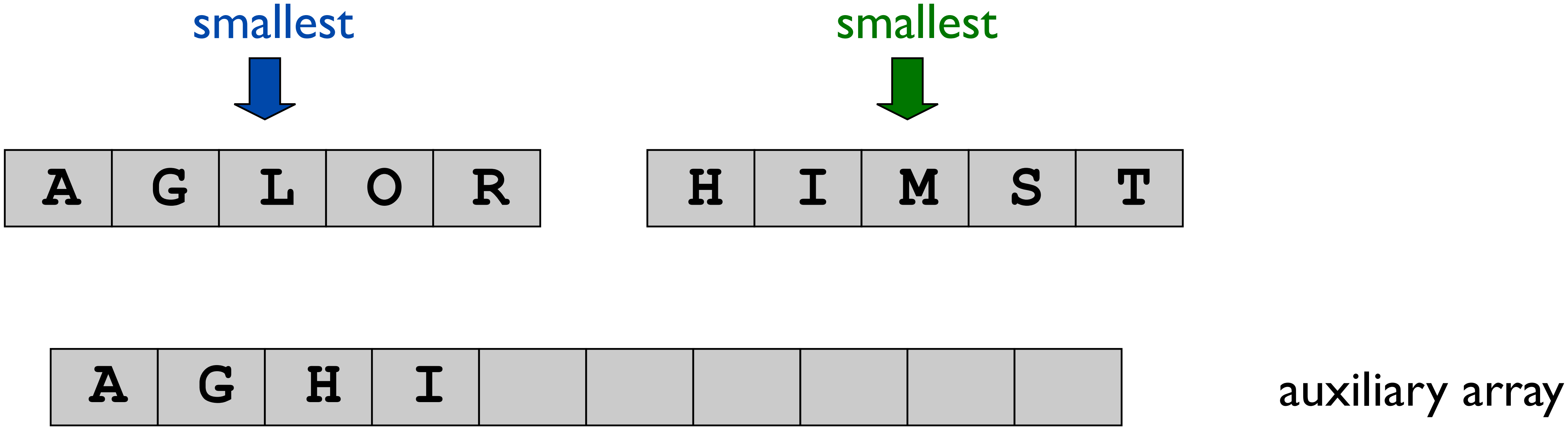

smallest

smallest

A G L O R

H I M S T

A G H I

auxiliary array

Merge

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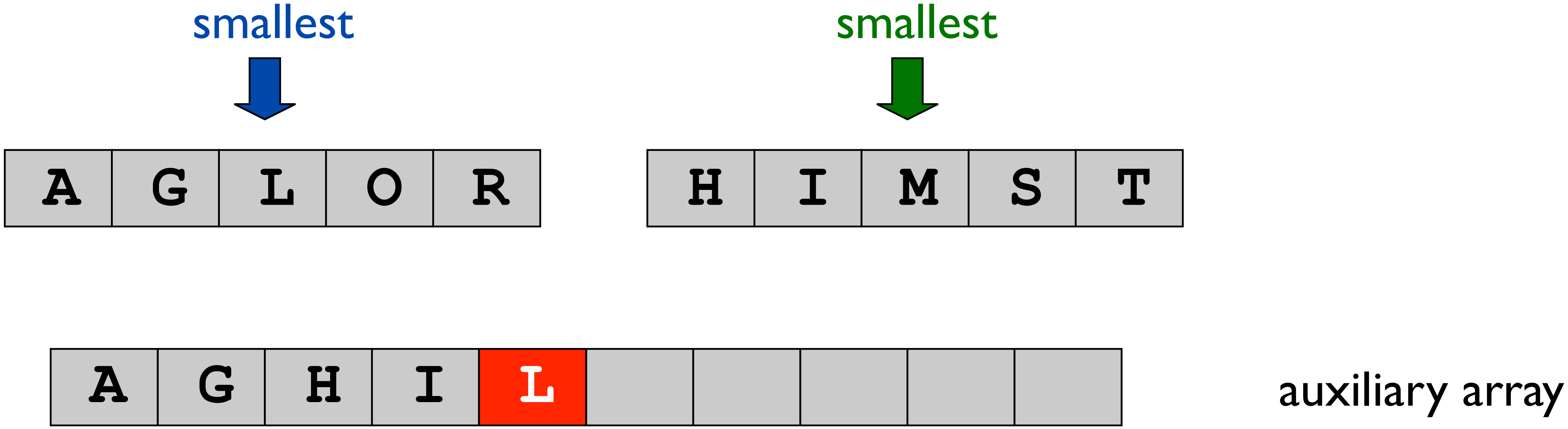

A	G	L	O	R

H	I	M	S	T

A	G	H	I	L					

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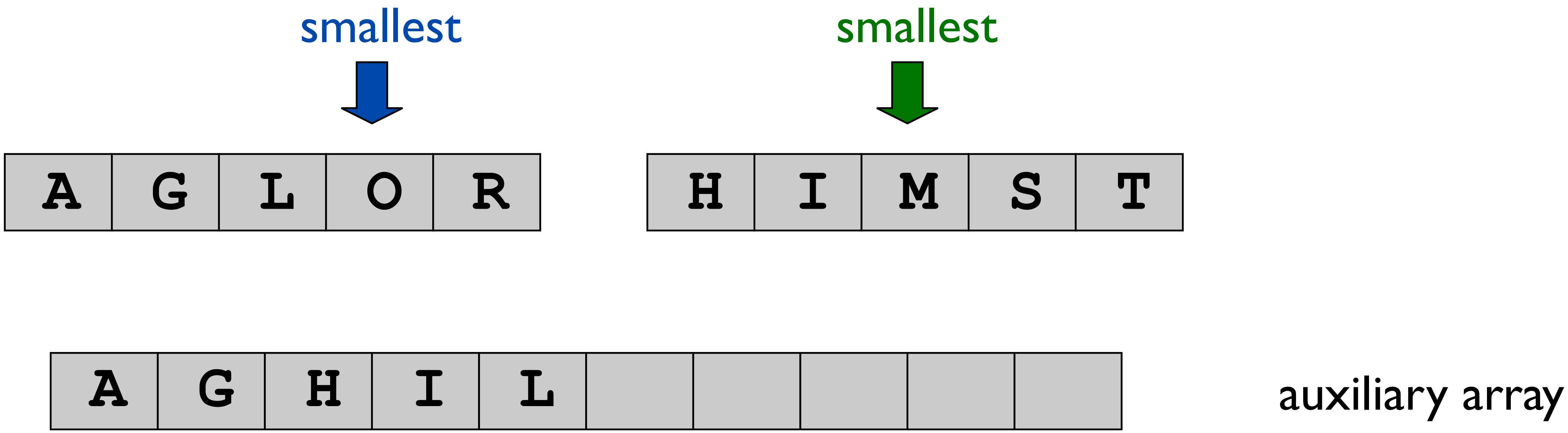

smallest

smallest

A G L O R

H I M S T

A G H I L

auxiliary array

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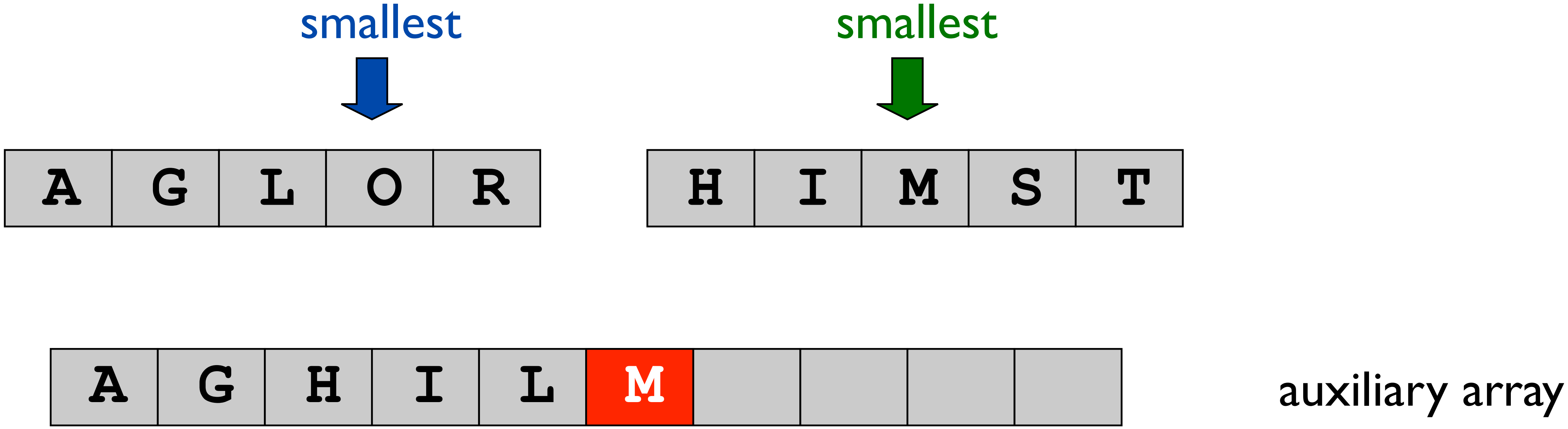

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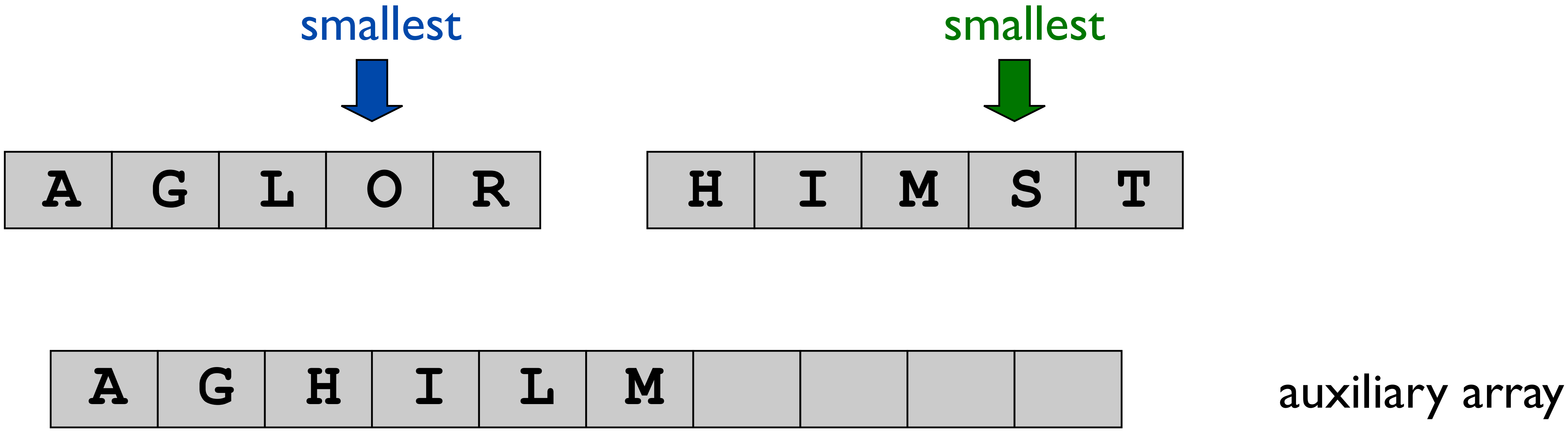

smallest

smallest

A	G	L	O	R

H	I	M	S	T

A	G	H	I	L	M				

auxiliary array

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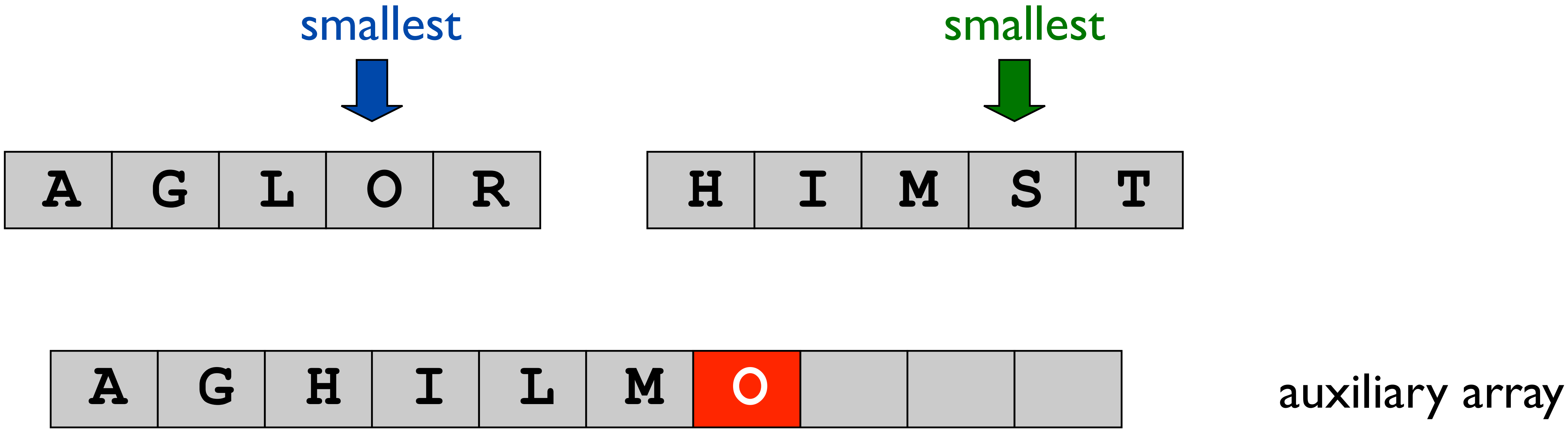

smallest

smallest

A G L O R

H I M S T

A G H I L M O

auxiliary array

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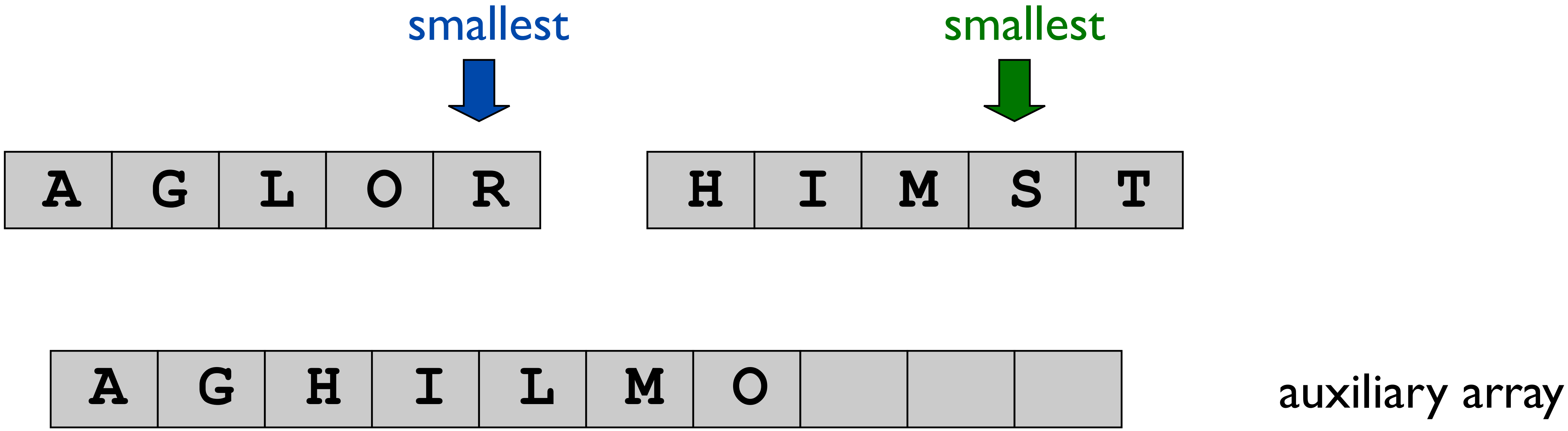

smallest

smallest

A	G	L	O	R

H	I	M	S	T

A	G	H	I	L	M	O			

auxiliary array

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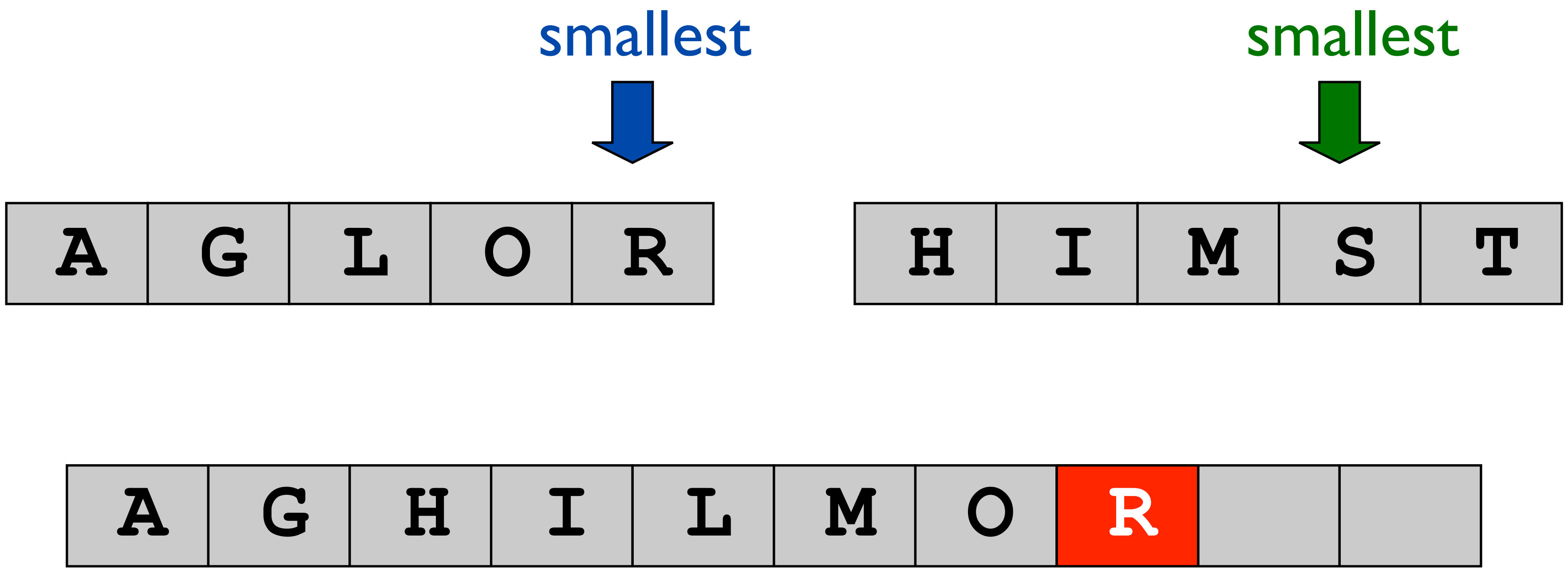

smallest

smallest

A	G	L	O	R

H	I	M	S	T

A	G	H	I	L	M	O	R		

auxiliary array

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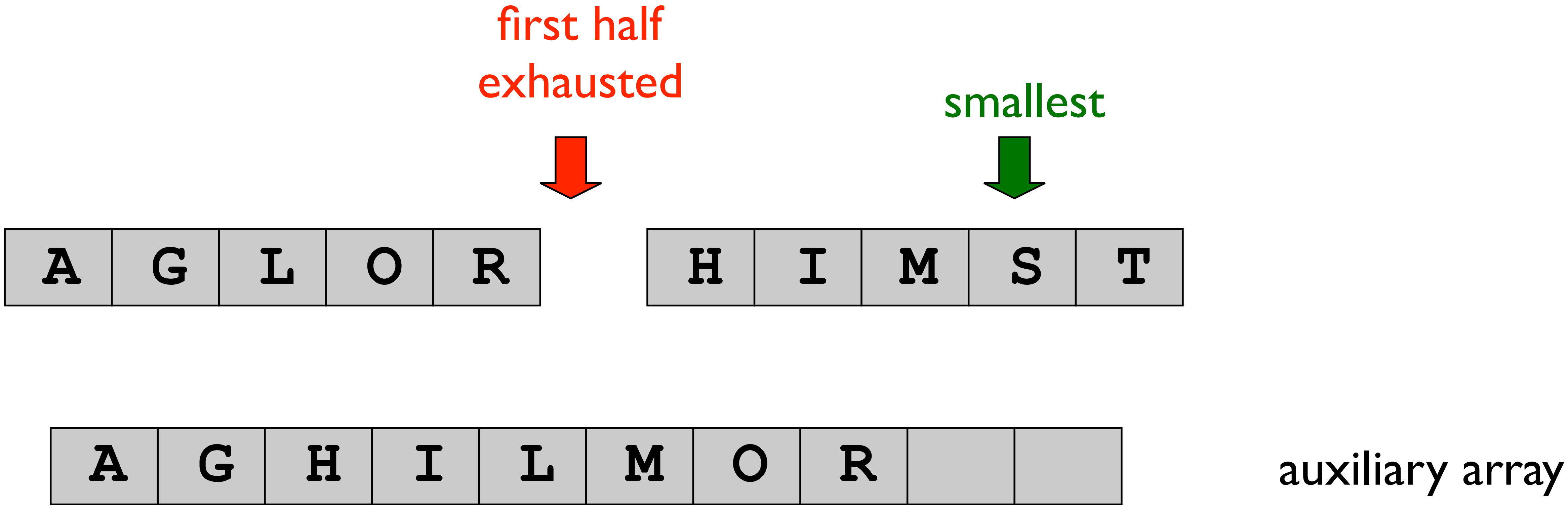


first half exhausted

smallest

A	G	L	O	R

H	I	M	S	T

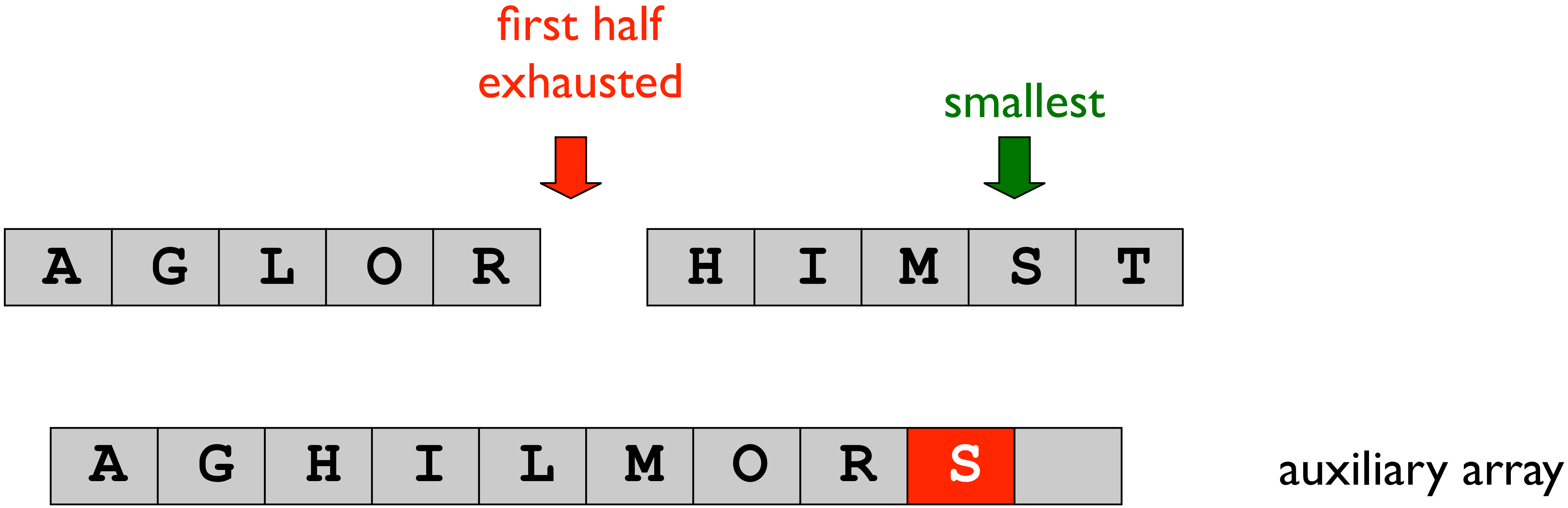
A	G	H	I	L	M	O	R		

auxiliary array

Merge

Merging.

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into auxiliary array.
- Repeat until done.



first half exhausted

smallest

A	G	L	O	R

H	I	M	S	T

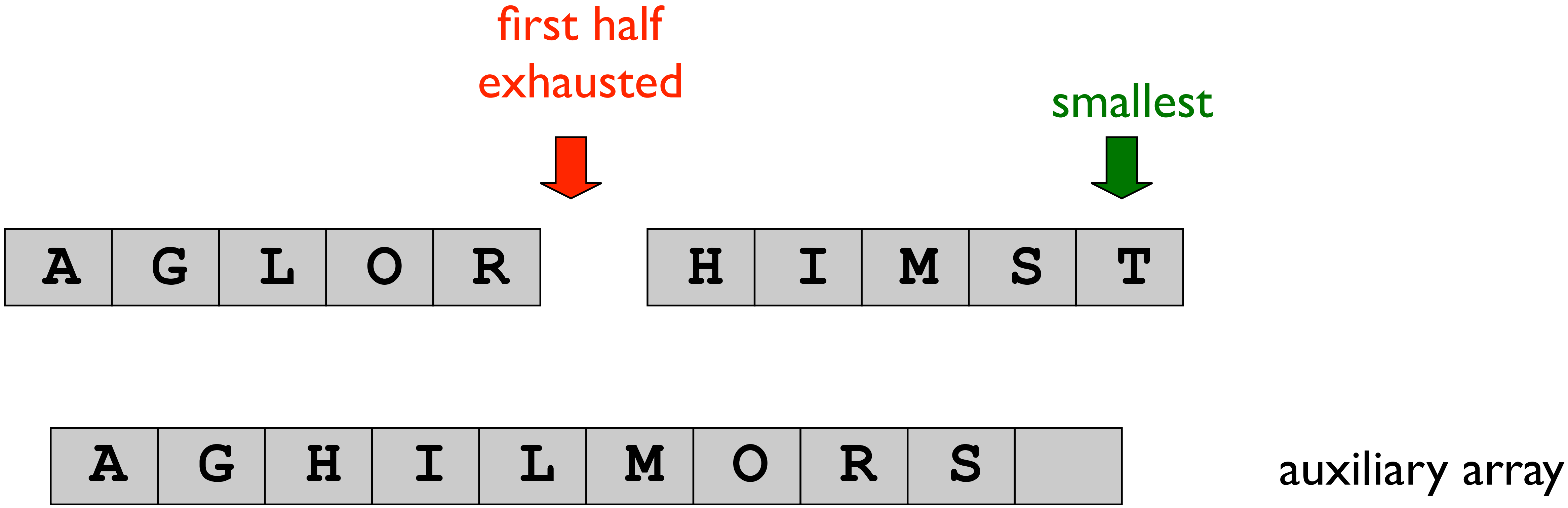
A	G	H	I	L	M	O	R	S	

auxiliary array

Merge

Merging.

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into auxiliary array.
- Repeat until done.



first half exhausted

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H	I	M	S	T

A	G	H	I	L	M	O	R	S	

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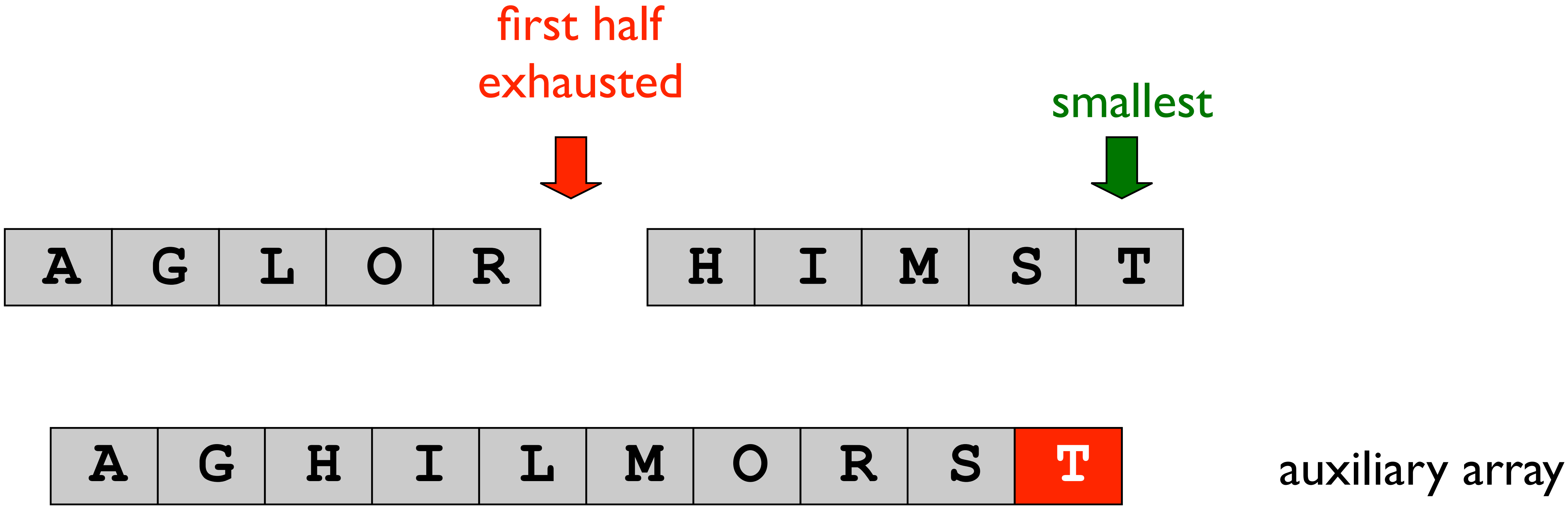

first half exhausted

smallest

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A	G	H	I	L	M	O	R	S	T

auxiliary array

Merge

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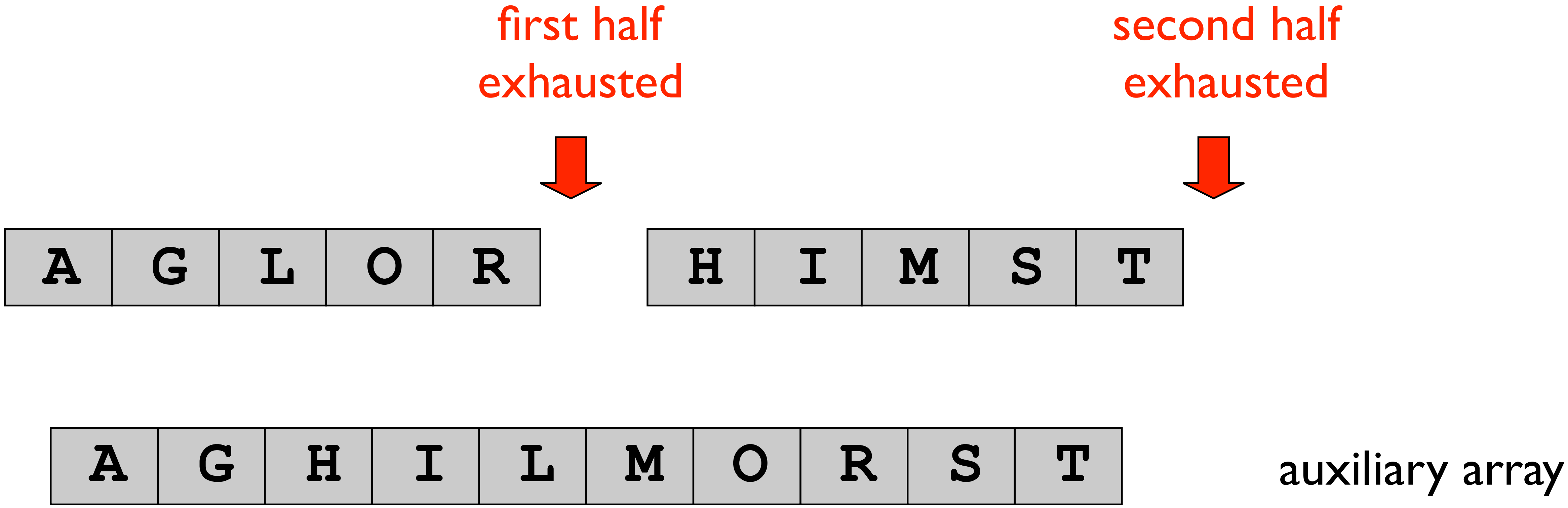
- Insert smallest of two elements into auxiliary array.
- Repeat until done.



first half exhausted

second half exhausted

A	G	L	O	R

H	I	M	S	T

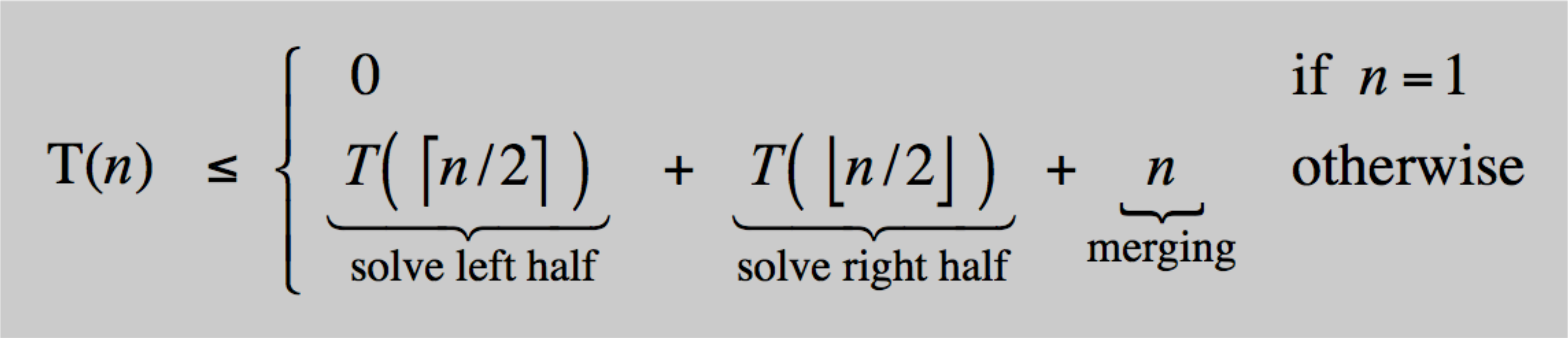
A	G	H	I	L	M	O	R	S	T

auxiliary array

Recurrence Relation

Def. T(n) = number of comparisons to mergesort an input of size n.

Mergesort recurrence.



$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

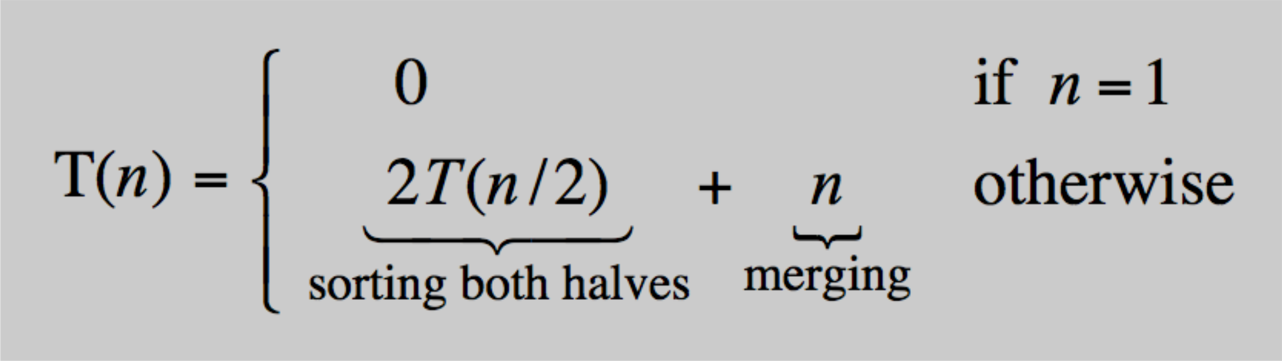
Solution. T(n) is $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace ≤ with =.

Telescoping Proof

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

assumes n is a power of 2



$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. For n > 1:

$$\begin{aligned}
\frac{T(n)}{n} &= \frac{2T(n/2)}{n} + 1 \\
&= \frac{T(n/2)}{n/2} + 1 \\
&= \frac{T(n/4)}{n/4} + 1 + 1 \\
&\cdots \\
&= \frac{T(n/n)}{n/n} + \underbrace{1 + \cdots + 1}_{\log_2 n} \\
&= \log_2 n
\end{aligned}$$

Induction Proof

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on k such that $n=2^k$)

- Base case: $n = 2^0 = 1$.
- Inductive hypothesis: $T(n) = T(2^k) = n \log_2 n$.
- Goal: show that $T(2n) = T(2^{k+1}) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2T(n) + 2n \\ &= 2n \log_2 n + 2n \\ &= 2n\left(\log_2 (2n) - 1\right) + 2n \\ &= 2n \log_2 (2n) \end{aligned}$$

Generalized Induction Proof

Claim. If T(n) satisfies the following recurrence, then $T(n) \le n \lceil \underset{\substack{\uparrow \\ \log_2 n}}{\lg} n \rceil$.

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: $n = 1$. $T(1) = 0 = 1 \lceil \lg 1 \rceil$.
- Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$. (note $1 \le n_1 < n$, $1 \le n_2 < n$)
- Induction step: Let $n \ge 2$, assume true for 1, 2, ... , n–1.

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ &\le n_1 \lceil \lg n_1 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &\le n_1 \lceil \lg n_2 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &= n \lceil \lg n_2 \rceil + n \\ &\le n(\lceil \lg n \rceil - 1) + n \\ &= n \lceil \lg n \rceil \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \left\lceil 2^{\lceil \lg n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \lg n \rceil} / 2 \\ \Rightarrow \lg n_2 &\le \lceil \lg n \rceil - 1 \end{aligned}$$

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COMP-250: Introduction to Computer Science

Lecture 10, February 11, 2016